1 Game theory [25 pts.]

i) Consider the following game:

	X	Y	Z
A	2, -2	0, 0	-3, 3
B	5, -5	-1, 1	-2, 2
C	3, -3	-1, 1	4, -4

(a) (4 points) Write down a linear program for computing a Nash equilibrium specifically for the game above.

(b) (1 point) Iteratively remove all (weakly) dominated pure strategies, if there are some strategies to remove.

(c) (4 points) Find a (possibly mixed) Nash equilibrium of this game. (Hint: you **do not** have to solve the linear program from task (a))

ii) (2 points) Imagine you have a general-sum game G and you formulate and solve a linear program as if this were a zero-sum game (you use utilities of the first player in the LP). What kind of solution concept of game G you computed (if any)?

iii) (3 points) Give an example of a game where the leader in a Stackelberg equilibrium receives a strictly lower expected utility compared to the expected utility of some Nash equilibrium or argue why such a game cannot exist.

iv) (2 points) Can there be a **two-player zero-sum game** with two different Nash equilibrium points (either pure or mixed) such that the expected value of player 1 is different in these points? Give an example of such a game, or provide a counterexample why such a game cannot exist.

v) Consider the following two player card game. The game begins with a mandatory bet of 1 chip for both players. After the bet, both players obtain 1 card randomly chosen from the deck of 3 cards containing one J, one K, and one Q. The players know their own card but do not observe the card of the opponent. After the cards are dealt, player 1 decides if he wants to continue playing by betting another chip or end the game by folding. If he decides to bet, player 2 also decides between betting 1 chip or folding. If both players decide to bet, the cards are revealed and the player with higher card wins all the chips currently in game. If either of players decides to fold, he immediately loses and all the chips currently in game are won by his opponent.

(a) (6 points) Formulate this game as an extensive-form game and write down the game tree, draw all information sets, and write down utilities for at least half of the leafs.

(b) (3 points) Write down all pure strategies of player 1.

2 Cooperative Game Theory [20 pts.]

i) (4 points) Three agents accept allocations $(2, 1, 0)$, $(0, 3, 0)$, or any convex combination of the two allocations. Find a three-player supermodular coalitional game whose core is the set of such accepted allocations. Explain how you obtained the solution.

ii) (4 points) Alice, Bob, and Cecilia are creditors of two bankrupt companies. The first and the second company's total estate is 100 and 200, respectively. The claims of Alice, Bob, and Cecilia to the first company are $200, 50, 100$ and to the second company are $50, 150, 50$. Describe the two bankruptcy games v and w. How would you compute the nucleolus of game $v + w$?

iii) (4 points) Decide if the following three-player coalitional game is supermodular or superadditive and explain your answer.:

$$v(1) = 1, v(2) = 2, v(3) = 0, \quad v(12) = 3, v(13) = v(23) = 1, \quad v(123) = 3.$$

Is the core nonempty? If it is true, describe all the core allocations.

iv) (8 points) Four shareholders own the following number of shares: $25, 16, 10, 49$. A binary vote is approved if the total number of affirmative votes is at least 66. The shareholder with 10 shares claims that his voting power – measures by the Banzhaf index – is the same as that of the shareholder having 16 shares. Is it true?

3 Auctions and Resource Allocation [15 pts.]

i) (5 points) Consider the following resource allocation problem: There are four resources $\{r_1, r_2, r_3, r_4\}$ and three agents $\{a, b, c\}$. The utility that a respective agent receives for being allocated a particular resource is given by the table below and we further assume the utilities are additive when allocated multiple resources.

	r_1	r_2	r_3	r_4
u_a	0	0	20	20
u_b	0	5	0	7
u_c	5	0	7	0

(a) Which would be an efficient allocation of resources to agents and what would be the resulting value of the respective social welfare function?

(b) Which would be a fair allocation of resources to agents and what would be the resulting value of the respective social welfare function?

(c) What is the price of fairness in resource allocation?

(d) What would be the price of (egalitarian) fairness in the above specified resource allocation problem?

ii) (5 points) Consider a single-item auction with three bidders $\{a, b, c\}$ which value the auctioned good with the following valutions: $v_a = 20, v_b = 25, v_c = 15$. Assume the auction is run using the *English* auction mechanism and that all bidders bid rationally.

(a) Describe how the auction could progress in terms of the sequence of actions carried out by the bidders and/or the auctioner.

(b) Which bidder would win the auction?

(c) Which payoff would each bidder receive from participating in the auction?

(d) Would the auction maximize the revenue the seller could get from selling the good? Justify your answer.

(e) Would the bidder payoffs change if the first-price sealed bid auction was used instead? Justify your answer.

iii) (5 points) Consider a combinatorial Vickerey-Clarke-Groves (VCG) auction with three bidders $\{a, b, c\}$, two goods $\{g_1, g_2\}$ and the following valuations of bundles:

bundle	$\{g_1\}$	$\{g_2\}$	$\{g_1, g_2\}$
v_a	2	3	4
v_b	4	2	5
v_c	1	1	6

Assume that both bidders bid rationally.

(a) What would be the allocation of goods resulting from the auction?

(b) What would be the payments the agents would make in the auction?

(c) Would the allocation of the goods be efficient?