FLOW GAMES

The description of concept for Assignment #3

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The purpose of these notes is to describe the concept of a flow game, which is a coalitional game derived from a flow problem in a network. Edges of the network are controlled by players and the worth of each coalition is determined by the maximum flow of the corresponding flow problem. See [1] for more details on flow games.

1 FLOW PROBLEMS

A *flow network* is a tuple (V, E, v^0, v^1, c), where

- (V, E) is a directed graph,
- $v^0 \in V$ is a *source* (no edges in),
- $v^1 \in V$ is a *sink* (no edges out),
- $c\colon E \to \mathbb{N}_0$ is a *capacity function*.

We can think of a flow network as a communication system between v^0 and v^1, in which each link $(u, v) \in E$ connecting vertices $u \in V$ and $v \in V$ has some capacity $c(u, v)$. The main goal is to transport as much of certain commodity (information, natural resources etc.) as possible from source v^0 to sink v^1, while not exceeding capacities of all links and obeying the conservation law "inflow = outflow". This is modelled by the notion of flow. A *flow* is a function $f\colon E \to \mathbb{N}_0$ such that:

- $f(u, v) \leq c(u, v)$ for all $(u, v) \in E$.
- For every vertex $v \in V \setminus \{v^0, v^1\}$,

$$\sum_{\substack{u \in V \setminus v \\ (u,v) \in E}} f(u, v) = \sum_{\substack{u \in V \setminus v \\ (v,u) \in E}} f(v, u).$$

The *magnitude* of a flow f is the total flow $M(f)$ arriving at v^1,

$$M(f) := \sum_{\substack{u \in V \setminus v^1 \\ (u,v^1) \in E}} f(u, v^1).$$

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It is also true that

$$M(f) = \sum_{\substack{u \in V \setminus v^0 \\ (v^0,u) \in E}} f(v^0, u).$$

The goal is to maximize the magnitude over the set of all flows. It is well-known that the flow whose magnitude is maximal exists and is called a *maximum flow*. Algorithms for computing a maximum flow are explained in the course Combinatorial Optimization. For the purposes of these notes it is enough to know that efficient algorithms to compute a maximum flow are available even for relatively large networks.

In addition to the data of a flow network, we assume that each edge in a network is controlled by a player from the player set $N := \{1, \dots, n\}$. This is captured by a function $I \colon E \to N$, where $I(u, v) = j$ means that edge $(u, v) \in E$ is controlled by player $j \in N$. A *flow problem* is then defined as a tuple

$$\mathcal{F} := (V, E, v^0, v^1, c, N, I). \tag{1}$$

2 FLOW GAMES

We will describe a coalitional game in which the worth of each coalition is the magnitude of a maximum flow that can be effectively transported only by the members of the coalition. Let $\mathcal{F}$ be a flow problem (1). For each coalition $A \subseteq N$, let E_A be the set of links controlled by the members of A,

$$E_A := \{(u, v) \in E \mid I(u, v) \in A\}.$$

Thus, we obtain a flow problem

$$\mathcal{F}_A := (V, E_A, v^0, v^1, c, A, I).$$

Note that the domains of mappings c and I above are restricted to E_A. A *flow game* is a coalitional game $v_{\mathcal{F}}$ over the player set N such that the worth of each $A \subseteq N$ is defined as

$$v_{\mathcal{F}}(A) := \text{the magnitude of a maximum flow for flow problem } \mathcal{F}_A.$$

In words, the number $v_{\mathcal{F}}(A)$ measures the maximal throughput from v^0 to v^1 using only the communication links under the control of coalition A. We mention several properties of every flow game $v_{\mathcal{F}}$.

- *Nonnegativity.* For every $A \subseteq N$, we have $v_{\mathcal{F}}(A) \geq 0$.
- *Superadditivity.* For every $A, B \subseteq N$ with $A \cap B = \emptyset$,

  $$v_{\mathcal{F}}(A \cup B) \geq v_{\mathcal{F}}(A) + v_{\mathcal{F}}(B).$$

- *Monotonicity.* For every $A, B \subseteq N$ with $A \supseteq B$,

  $$v_{\mathcal{F}}(A) \geq v_{\mathcal{F}}(B).$$

3 VALUES OF PLAYERS IN FLOW GAME

What is the influence of players on the amount of commodity transfered from the source to the sink in a flow game $v_{\mathcal{F}}$? The influence can be described by value operators such as the Shapley value or the Banzhaf value. We will briefly summarize basic facts about these concepts, which were explained in the lectures on coalitional games.

The *Shapley value* of player $i \in N$ in flow game $v_{\mathcal{F}}$ can be presented as

$$\varphi_i^S(v_{\mathcal{F}}) = \sum_{\pi \in \Pi} \frac{1}{n!} \cdot x_i^{\pi}, \tag{2}$$

where $\boldsymbol{x}^{\pi}$ is a marginal vector for flow game $v_{\mathcal{F}}$ and permutation π. We can formulate a straightforward sampling procedure to estimate $\varphi_i^S(v_{\mathcal{F}})$ by a sample mean – see the lectures on coalitional games for details. We note that an equivalent expression for the Shapley value is

$$\varphi_i^S(v_{\mathcal{F}}) = \sum_{A \subseteq N \setminus i} \frac{|A|!(n-|A|-1)!}{n!} \cdot (v_{\mathcal{F}}(A \cup i) - v_{\mathcal{F}}(A)). \tag{3}$$

The *Banzhaf value* $\varphi_i^B(v_{\mathcal{F}})$ of player $i \in N$ in flow game $v_{\mathcal{F}}$ is defined as

$$\varphi_i^B(v_{\mathcal{F}}) = \sum_{A \subseteq N \setminus i} \frac{1}{2^{n-1}} \cdot (v_{\mathcal{F}}(A \cup i) - v_{\mathcal{F}}(A)). \tag{4}$$

The formulas (3) and (4) differ only in the probabilities used in the corresponding expected values. Computing (4) is not tractable for large flow games and sampling is the only option. The value $\varphi_i^B(v_{\mathcal{F}})$ can be estimated by a sample mean computed by randomly sampling coalitions $A \subseteq N \setminus i$ from a uniform distrubution over $\mathcal{P}(N \setminus i)$. Specifically, when m is the size of such a random sample and $A_1, \ldots, A_m \in \mathcal{P}(N \setminus i)$ are the sampled coalitions, the estimate of the Banzhaf value is

$$\sum_{k=1}^{m} \frac{1}{m} \cdot (v_{\mathcal{F}}(A_k \cup i) - v_{\mathcal{F}}(A_k)).$$

3.1 Small example

This flow game comes from [1, Example 17.65]. The flow problem $\mathcal{F}$ over the player set $N = \{1, 2, 3\}$ is depicted in Figure 1. In the picture, each directed link is labelled with a capacity and a player who is controlling the link. For example, label "2, 1" means that the corresponding link has capacity 2 and it is under the control of player 1.

The associated flow game is

$$v_{\mathcal{F}}(A) = \begin{cases} 0 & A = \emptyset, 1, 2, 3, 23, \\ 2 & A = 12, 13, \\ 4 & A = N. \end{cases}$$

The Shapley values of players are $(2, 1, 1)$. In this case the Banzhaf values are the same.

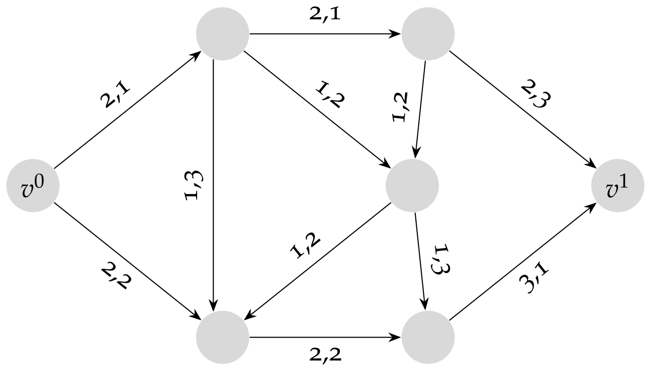

Figure 1: Small flow problem

3.2 Large example

Large flow games are computationally intractable. The main limiting factor is the number of players n. As the Shapley value (2) averages marginal vectors over all permutations, its expected asymptotic complexity is factorial. Figure 2 shows that computing the Shapley value is indeed the bottleneck. To put the growth rate into perspective, if our algorithm could compute a flow game with 10 players in just 1 ms, a similar game with 20 players would take 21 years.

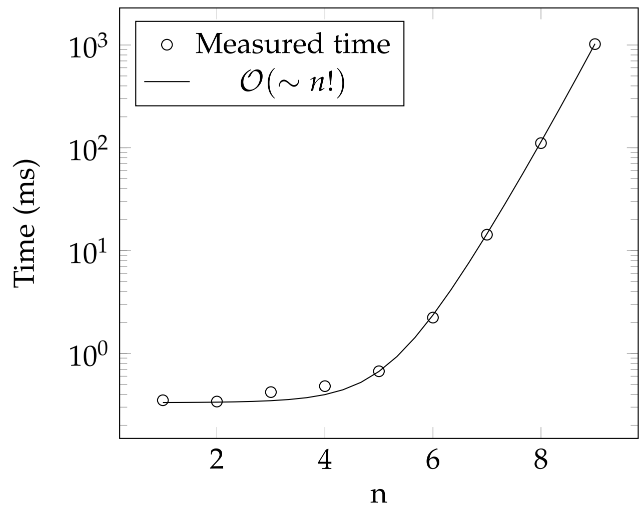

Figure 2: Measured runtime vs Stirling's approximation of $n!$ (taking into account the 350*ms* startup time of *Gurobi*) for flow games with n players

The logical conclusion is that if we keep the number of agents low, we can afford to process relatively large flow problems. A case in point is the flow game with 777 vertices, 1241 edges, and 9 players in Figure 3. The Shapley and the Banzhaf values of players rounded to two decimal places are in Table 1. Table 2 depicts the ranking $\succ$ of the players according to these values. The ranking proceeds from the most to the least influential player and the players with equal value are separated by a comma.

Shapley	1.16	1.14	1.55	0.94	1.07	0.96	0.87	1.55	0.75
Banzhaf	0.17	0.16	0.20	0.12	0.13	0.13	0.14	0.20	0.13

Table 1: Influence of players

Shapley	$3,8 \succ 1 \succ 2 \succ 5 \succ 6 \succ 4 \succ 7 \succ 9$
Banzhaf	$3,8 \succ 1 \succ 2 \succ 7 \succ 6,5,9 \succ 4$

Table 2: Ranking of players

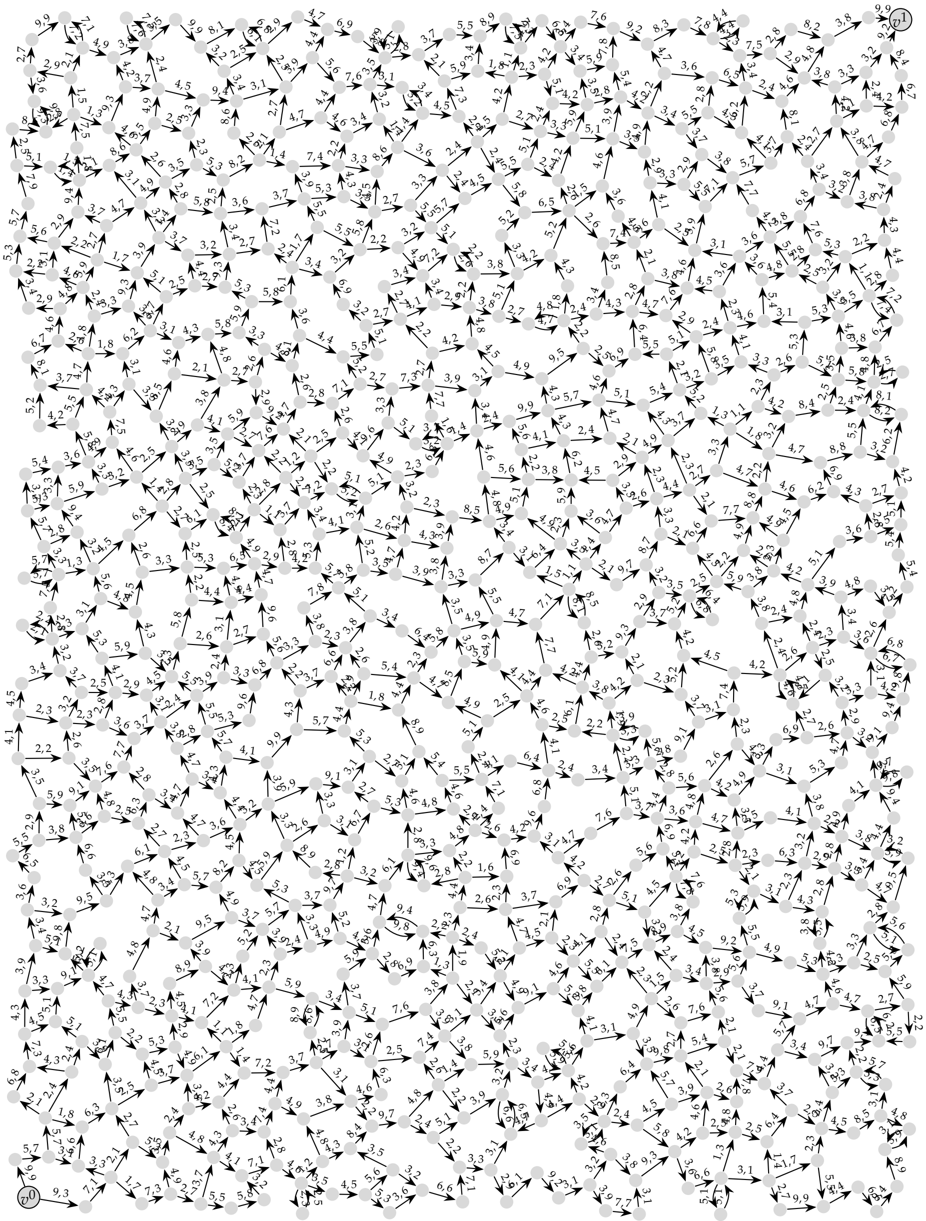

Figure 3: Large flow problem

REFERENCES

[1] M. Maschler, E. Solan, and S. Zamir. *Game Theory*. Cambridge University Press, 2013.