COMPUTATIONAL GAME THEORY

Exercises

TOMÁŠ KROUPA*

September 20, 2022

CONTENTS

* *AI Center, Department of Computer Science, Faculty of Electrical Engineering, Czech Technical University in Prague*

ABOUT THIS DOCUMENT

The aim of these exercises is to make you, the student of *Computational Game Theory*, think actively about the concepts introduced in the lectures. You are encouraged to work out your approach to the solution. Almost all exercises can be solved using pen & paper – write your notes and calculations on the wide margins of this document. Do not hesitate to consult the course materials available online for the basic notions and results. Most of the exercises are adapted from [1, 2, 3, 4]. Some questions are more difficult or require more extended mathematical arguments. Such items are marked with ★ .

In addition to the exercises covering the content of the lectures, the first section presents selected mathematical prerequisities necessary for understanding game theory. This is mostly based on the main concepts discussed in the undergraduate courses of Linear Algebra, Optimization, Linear Programming, Discrete Mathematics, and Probability Theory.

1 MATHEMATICAL PREREQUISITIES

Exercise 1.
Duality in linear programming. The concept of duality appears in many game-theoretic problems. In this exercise we consider the linear program

$$\begin{aligned} \text{Minimize} \quad & -x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 4 \\ & 2x_1 + x_2 \leq 5 \\ & -x_1 + 4x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Characterize the feasible set of the problem by its vertices (extreme points). Further, find the optimal solution of this linear program and formulate the dual program. What is the meaning of the dual program and its relation to the original program? What will be the optimal value of the dual problem?

Exercise 2.
Minimax/maximin values. Let X and Y be nonempty sets and consider a real function $f\colon X \times Y \to \mathbb{R}$. Show that

$$\max_{x \in X} \min_{y \in Y} f(x,y) \leq \min_{y \in Y} \max_{x \in X} f(x,y), \tag{1}$$

where we assume that all maxima and minima above exist. Find an example of function f taking at least two different values such that (a) the inequality is strict and (b) the inequality becomes an equality. *Hint:* Such examples exist already when both X and Y have 2 elements.

Exercise 3.
Saddle points. Let $f\colon X \times Y \to \mathbb{R}$ be a function, where X and Y are arbitrary nonempty set. We say that $(x^*, y^*) \in X \times Y$ is a *saddle point* of f if

$$f(x, y^*) \leq f(x^*, y^*) \leq f(x^*, y) \qquad \text{for all } x \in X,\ y \in Y. \tag{2}$$

Show that a function f has a saddle point if, and only if,

$$\max_{x \in X} \min_{y \in Y} f(x,y) = \min_{y \in Y} \max_{x \in X} f(x,y), \tag{3}$$

provided that all the maxima/minima above exist.

Exercise 4.
Joining a random coalition. Assume that $N = \{1, \ldots, n\}$ is a set of players. A *coalition* is any subset of N. One of the players, say $i \in N$, would like to join a coalition A of other players. What is the probability of selecting A at random? What is the probability that a coalition A is chosen in the following way. First, player i randomly picks the size of A, and then A is selected among the coalitions of such size at random?

SOLUTIONS

Solution 1.
In any linear programming problem, the feasible set is a convex polyhedron. In our case the polyhedron is bounded, so it is a convex polygon characterized by its vertices. Every vertex corresponds to a unique solution of some linear equality system associated with a subset of linear inequality constraints of the problem. For example, a point $(1,3)$ is a vertex since it is a unique solution to linear equations $x_1 + x_2 = 4$ and $2x_1 + x_2 = 5$. The remaining vertices are $(0, \frac{1}{2}), (0,4), (2,1)$. See Figure 1, which is an output of the simple online solver `https://online-optimizer.appspot.com`.

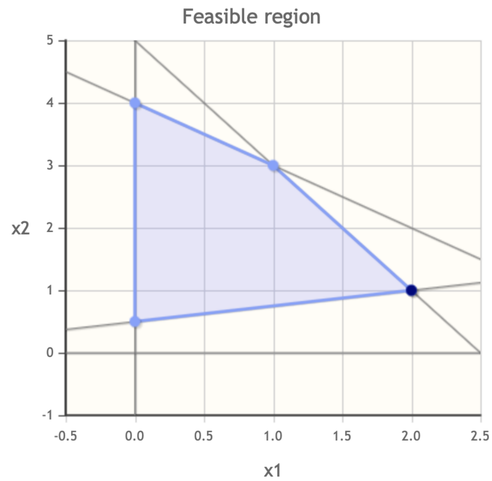

Figure 1: The feasible set corresponding to constraints (5)–(7)

What is the optimal solution to the problem? The fundamental theorem of linear programming says that, if a linear program has an optimal solution, then it is attained at some vertex of the feasible set. Clearly, at least one optimal solution exists in our case since the feasible set is bounded. Therefore, it suffices to compute the values of the objective function $f(x_1, x_2) = -x_1 + 2x_2$ for all the vertices and select the one with the minimal value. This yields the optimal solution $(2,1)$ with the optimal value is $f(2,1) = 0$.

To formulate the dual problem, it is convenient to formulate the original problem so that all inequalities are of the form $\geq$:

$$\text{Minimize} \quad -x_1 + 2x_2 \tag{4}$$
$$\text{subject to} \quad -x_1 - x_2 \geq -4 \tag{5}$$
$$-2x_1 - x_2 \geq -5 \tag{6}$$
$$-x_1 + 4x_2 \geq 2 \tag{7}$$
$$x_1, x_2 \geq 0 \tag{8}$$

By design, the dual program has 3 variables y_1, y_2, y_3 corresponding to the 3 primal linear constraints (5)–(7), and 2 linear constraints associated with

2 primal variables x_1, x_2. The exact form of the dual program can be found in any textbook on linear programming. Specifically the dual is

$$\begin{aligned} \text{Maximize} \quad & -4y_1 - 5y_2 + 2y_3 \\ \text{subject to} \quad & -y_1 - 2y_2 - y_3 \leq -1 \\ & -y_1 - y_2 + 4y_3 \leq 2 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

The duality theorem immediately implies that the primal and dual problems have the same optimal value 0.

What is the interpretation of duality? The optimal value of dual problem provides the tighest lower bound on the solution of the primal problem. In fact, the dual problem controls the optimal value of the primal problem from below, and does so in the best possible way — the optimal values of both programs coincide!

We will elaborate on the idea sketched above. How to obtain a lower bound for the primal problem? Note that one possible lower bound on the values of the objective f follows from the nonnegativity constraints (8) and the constraint (5) since

$$-4 \leq -x_1 - x_2 \leq -x_1 + 2x_2.$$

Can we do better? That is, can we obtain a higher value than -4 for the lower bound? It is easy to see that by multiplying the inequalities (5) and (7) by $\frac{1}{2}$ and adding them we obtain the bound

$$-1 \leq -x_1 + \tfrac{3}{2}x_2 \leq -x_1 + 2x_2,$$

which is tighter. Can we still improve on this lower bound? Let us try to generalize the idea of combining the linear inequalities of the primal problem for estimating the optimal value from below. The goal is to find nonnegative real numbers y_1, y_2, y_3 such that

$$-4y_1 - 5y_2 + 2y_3 \leq \underbrace{y_1(-x_1 - x_2) + y_2(-2x_1 - x_2) + y_3(-x_1 + 4x_2)}_{(-y_1-2y_2-y_3)x_1+(-y_1-y_2+4y_3)x_2} \leq -x_1 + 2x_2$$

and the second inequality should be as tight as possible. One way to guarantee this is to maximize the most left-hand side $-4y_1 - 5y_2 + 2y_3$ while preserving the constraints $-y_1 - 2y_2 - y_3 \leq -1$ and $-y_1 - y_2 + 4y_3 \leq 2$ expressed by the second inequality above. In other words, we have derived precisely the dual problem.

Solution 2.
To prove the inequality (1), define

$$F(x) = \min_{y \in Y} f(x, y), \qquad x \in X,$$

and let $x^* \in X$ be the maximizer of F over X. Then

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \max_{x \in X} F(x) = F(x^*) = \min_{y \in Y} f(x^*, y).$$

For every $y \in Y$, we have

$$f(x^*, y) \leq \max_{x \in X} f(x, y)$$

by the definition of maximum, which implies

$$\min_{y \in Y} f(x^*, y) \leq \min_{y \in Y} \max_{x \in X} f(x, y).$$

Hence, (1) is proved.

We will construct two examples (a)-(b) under the assumption that $X = Y = \{1, 2\}$. First, consider a function f given by the matrix

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{Y} \\ & & 1 & 2 \\ \cline{3-4} X & 1 & 0 & 1 \\ \cline{3-4} & 2 & 1 & 0 \\ \cline{3-4} \end{array}$$

Then

$$\max_{x \in X} \min_{y \in Y} f(x, y) = 0 < 1 = \min_{y \in Y} \max_{x \in X} f(x, y).$$

For the second example (b), take a matrix

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{Y} \\ & & 1 & 2 \\ \cline{3-4} X & 1 & 3 & 0 \\ \cline{3-4} & 2 & 2 & 2 \\ \cline{3-4} \end{array}$$

Then

$$\max_{x \in X} \min_{y \in Y} f(x, y) = 2 = \min_{y \in Y} \max_{x \in X} f(x, y).$$

Solution 3.
We define two functions: $F(x) = \min_{y \in Y} f(x, y)$, for all $x \in X$, and $G(y) = \max_{x \in X} f(x, y)$, for every $y \in Y$. Assume that f has a saddle point (x^*, y^*). Then the saddle point definition (2) and the definition of maxima/minima imply

$$\min_{y \in Y} G(y) \leq G(y^*) = \max_{x \in X} f(x, y^*) = f(x^*, y^*) = \min_{y \in Y} f(x^*, y) = F(x^*) \leq \max_{x \in X} F(x).$$

We obtained the inequality

$$\min_{y \in Y} \max_{x \in X} f(x, y) \leq \max_{x \in X} \min_{y \in Y} f(x, y),$$

and since the converse inequality always holds (Exercise 2), we proved the minimax equality (3).

To prove the converse implication, assume that (3) holds. Let x^* and y^* be such that $F(x^*) = \max_{x \in X} F(x)$ and $G(y^*) = \min_{y \in Y} G(y)$, respectively. Then the assumption (3) gives

$$F(x^*) = \max_{x \in X} F(x) = \max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y) = \min_{y \in Y} G(y) = G(y^*). \tag{9}$$

Further, from the definition of F, G and x^*, y^* we get

$$F(x^*) \leq f(x^*, y^*) \leq G(y^*)$$

and (9) implies that even $F(x^*) = f(x^*, y^*) = G(y^*)$. Let $x \in X$ and $y \in Y$. Then

$$f(x, y^*) \leq G(y^*) = f(x^*, y^*) = F(x^*) \leq f(x^*, y).$$

Solution 4.
There are 2^{n-1} coalitions A to which player i doesn't belong, $i \notin A$. Therefore $p(A) = 2^{1-n}$ is the probability of picking such a coalition randomly. Consider now the second variant of random choice of the coalition. Let A be a random coalition selected in this way. We can compute the probability $q(A)$ using the definition of conditional probability,

$$q(A) = q(A \mid \text{the size is } |A|) \cdot q(\text{the size is } |A|) = \frac{1}{\binom{n-1}{|A|}} \cdot \frac{1}{n},$$

where we used the fact that $q(A) = q(A \wedge \text{the size is } |A|)$. We can easily verify that q is a probability distribution:

$$\sum_{A \subseteq N \setminus \{i\}} q(A) = \frac{1}{n} \sum_{A \subseteq N \setminus \{i\}} \frac{1}{\binom{n-1}{|A|}} = \frac{1}{n} \sum_{a=0}^{n-1} \binom{n-1}{a} \frac{1}{\binom{n-1}{a}} = 1.$$

Note that both p and q depend only on the size of each coalition. Specifically, if A and B are coalitions such that $|A| = |B|$, then $p(A) = p(B) = 2^{1-n}$ and $q(A) = q(B)$. Probability distributions p and q appear in the Banzhaf and Shapley values in coalitional game theory. We can see that there is a several order magnitude difference between their values (Figure 2). For example, in case that there are $n = 60$ players, the probabilities of coalition A with $|A| = 10$ are $p(A) \approx 1.7 \times 10^{-18}$ and $q(A) \approx 2.6 \times 10^{-13}$, respectively.

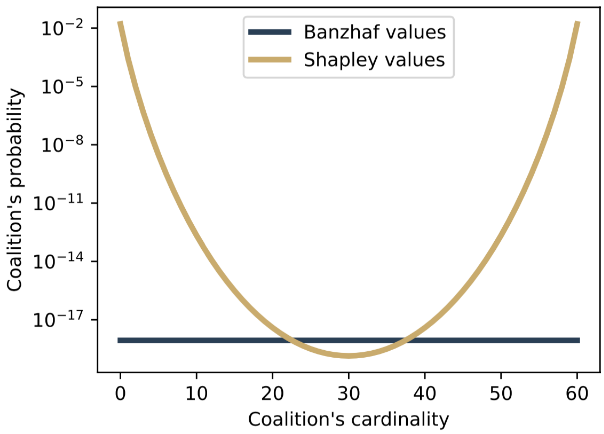

Figure 2: Values of p and q.

REFERENCES

[1] J. González-Díaz, I. García-Jurado, and M. G. Fiestras-Janeiro. *An Introductory Course on Mathematical Game Theory*, volume 115 of *Graduate Studies in Mathematics*. American Mathematical Society, 2010.

[2] M. Maschler, E. Solan, and S. Zamir. *Game Theory*. Cambridge University Press, 2013.

[3] G. Owen. *Game theory*. Academic Press Inc., San Diego, CA, third edition, 1995.

[4] B. Peleg and P. Sudhölter. *Introduction to the theory of cooperative games*, volume 34 of *Theory and Decision Library. Series C: Game Theory, Mathematical Programming and Operations Research*. Springer, Berlin, second edition, 2007.