COMPUTATIONAL GAME THEORY

Exercises

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1 NORMAL-FORM GAMES

Exercise 1.
Alice and Bob play Rock–Paper–Scissors, but Bob's fingers hurt preventing him from signalling "Scissors". Model this scenario as a zero-sum game and find its equilibrium.

Exercise 2.
We present the game called *Battle of the Sexes*. Its name is derived from the situation where a couple (Alice and Bob) is trying to plan what to do on Saturday. The alternatives are going to a concert (C) or watching a football match (F). Bob prefers football and Alice prefers the concert, but both prefer being together to being alone, even if that means agreeing to the less-preferred recreational activity.

		Bob	
		C	F
Alice	C	$2, 1$	$0, 0$
	F	$0, 0$	$1, 2$

Find equilibrium strategies of this game.

Exercise 3.
Consider a two-person zero-sum game with the payoff matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad a, b, c, d \in \mathbb{R}.$$

Find equilibrium strategies of the row and column player.

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Exercise 4.
Show that the following two-player zero-sum game doesn't have an equilibrium in pure strategies. The strategy space of each player is the set $X = \{0, \frac{1}{10}, \dots, \frac{9}{10}, 1\}$ and the payoff function of Player 1 is

$$u(s_1, s_2) = \frac{1}{1 + (s_1 - s_2)^2}, \qquad s_1, s_2 \in X.$$

Exercise 5.
Consider following utility matrix two-person normal-form game, where row player is a leader, which publicly announces its strategy

		F				
		a	*b*	*c*	*d*	*e*
L	*T*	2,4	6,4	9,0	1,2	7,4
	B	8,4	0,4	3,6	1,5	0,0

Find Strong and Weak Stackelberg Equilibrium

Exercise 6.
The two-player zero-sum game with the payoff matrix for the first player

		Player 2	
		0	1
Player 1	0	1	−1
	1	−1	1

is called *Matching Pennies*. In this game, each player chooses one bit (or a side of the coin), 0 or 1, in the following way: each player inserts into an envelope a slip of paper on which his choice is written. The envelopes are sealed and submitted to a referee. If both players have selected the same bit, Player 2 pays one dollar to Player 1. If they have selected opposite bits, Player 1 pays one euro to Player 2. Find an equilibrium of this game.

SOLUTIONS

Solution 1.
This is a matrix game which can be described by the payoff matrix for Alice:

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Since this is a two-person zero-sum game, its solution can be recovered by two dual linear programming problems. For Alice we solve the problem with variables x_0 and $\mathbf{x} = (x_1, x_2, x_3)^\intercal$:

$$\begin{aligned} \text{Maximize} \quad & x_0 \\ \text{subject to} \quad & \mathbf{A}^\intercal \mathbf{x} - \mathbf{1} x_0 \geq \mathbf{0}, \\ & \sum_{i=1}^{3} x_i = 1, \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

The problem for Bob has variables y_0 and $\mathbf{y} = (y_1, y_2)^\intercal$:

$$\begin{aligned} \text{Minimize} \quad & y_0 \\ \text{subject to} \quad & \mathbf{A}\mathbf{y} - \mathbf{1} y_0 \leq \mathbf{0}, \\ & \sum_{i=1}^{2} y_i = 1, \\ & \mathbf{y} \geq \mathbf{0}. \end{aligned}$$

Equivalently, we can write:

$$\begin{aligned} \text{Maximize} \quad & x_0 \\ \text{subject to} \quad & x_2 - x_3 - x_0 \geq 0, \\ & -x_1 + x_3 - x_0 \geq 0, \\ & x_1 + x_2 + x_3 = 1, \\ & x_1, x_2, x_3 \geq 0, \end{aligned}$$

and

$$\begin{aligned} \text{Minimize} \quad & y_0 \\ \text{subject to} \quad & -y_2 - y_0 \leq 0, \\ & y_1 - y_0 \leq 0, \\ & -y_1 + y_2 - y_0 \leq 0, \\ & y_1 + y_2 = 1, \\ & y_1, y_2 \geq 0. \end{aligned}$$

The solutions of those two problems are

$$\mathbf{x}^* = (0, \tfrac{1}{3}, \tfrac{2}{3})^\intercal \quad \text{and} \quad \mathbf{y}^* = (\tfrac{1}{3}, \tfrac{2}{3})^\intercal.$$

The value of game is equal to the common value in the optima, $x_0^* = y_0^* = \frac{1}{3}$.

Solution 2.

It is easy to verify that both strategy profiles (C, C) and (F, F) are equilibria in pure strategies. We will show that the game has a mixed strategy equilibrium, too.

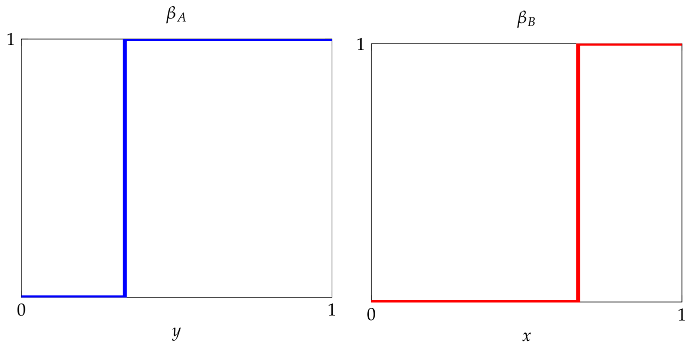
Let $x \in [0, 1]$ and $y \in [0, 1]$ be the probabilities of playing C for Alice and Bob, respectively. Since each player has only two pure strategies, the set of all mixed strategies can be viewed as the unit interval $[0, 1]$. Thus, the expected utility of Alice is $U_A(x, y) = 2xy + (1 - x)(1 - y)$ and the expected utility of Bob is $U_B(x, y) = xy + 2(1 - x)(1 - y)$, for all $x, y \in [0, 1]$.

Now, we compute the best responses of both players. For Alice this is the mapping defined by

$$\beta_A(y) = \arg\max_{x \in [0,1]} U_A(x, y), \qquad y \in [0, 1],$$

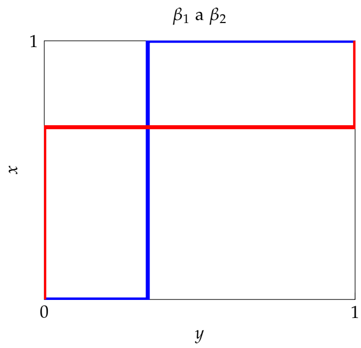
and analogously for Bob. We get

$$\beta_A(y) = \begin{cases} 0 & 0 \le y < \frac{1}{3}, \\ [0,1] & y = \frac{1}{3}, \\ 1 & \frac{1}{3} < y \le 1, \end{cases} \qquad \beta_B(x) = \begin{cases} 0 & 0 \le x < \frac{2}{3}, \\ [0,1] & x = \frac{2}{3}, \\ 1 & \frac{2}{3} < x \le 1. \end{cases}$$



We know that $(x^*, y^*) \in [0, 1]^2$ would correspond to an equilibrium in mixed strategies if, and only if,

$$x^* \in \beta_A(y^*) \ \text{ and } \ y^* \in \beta_B(x^*). \tag{1}$$



The geometric interpretation of the condition (1) is that $(x^*, y^*) \in [0, 1]^2$ is the point of the common intersection of the graphs of β_A a β_B. Therefore we obtain a mixed strategy equilibrium in which Alice plays the mixed strategy $(\frac{2}{3}, \frac{1}{3})$ and Bob uses the mixed strategy $(\frac{1}{3}, \frac{2}{3})$.

Solution 3.

Let $\mathbf{x} = (x, 1-x)^\intercal$ a $\mathbf{y} = (x, 1-y)^\intercal$ be the vectors of mixed strategies for the row and column player, respectively, where $x, y \in [0,1]$. Note that each such vector is fully determined by its first coordinate since every player has only two strategies. The expected payoff of the row player is then given by the function $U\colon [0,1]^2 \to \mathbb{R}$ such that

$$U(x,y) = \mathbf{x}^\intercal \mathbf{A}\mathbf{y} = (a+d-b-c)xy + (b-d)x + (c-d)y + d.$$

By von Neumann's minimax theorem, an equilibrium mixed strategy profile exists, and it necessarily corresponds to the saddle point (x^*, y^*) of U. By elementary analysis, we know that (x^*, y^*) must satisfy the condition

$$\frac{\partial U}{\partial x}(x^*, y^*) = \frac{\partial U}{\partial y}(x^*, y^*) = 0,$$

which reads as

$$(a+d-b-c)y + b - d = (a+d-b-c)x + c - d = 0.$$

Assume that $a+d-b-c \neq 0$. Then the only solution is

$$\begin{aligned} \mathbf{x}^* &= (x^*, 1-x^*)^\intercal = \tfrac{1}{a+d-b-c}(d-c, a-b)^\intercal, \\ \mathbf{y}^* &= (y^*, 1-y^*)^\intercal = \tfrac{1}{a+d-b-c}(d-b, a-c)^\intercal. \end{aligned}$$

Since the Hessian of U

$$\begin{bmatrix} 0 & a+d-b-c \\ a+d-b-c & 0 \end{bmatrix}$$

is indefinite, (x^*, y^*) is indeed the saddle point.

Now, let $a+d-b-c = 0$. The function U becomes

$$U(x,y) = (b-d)x + (c-d)y + d.$$

First, suppose $b \geq d$ and $c \geq d$. Since $a = b + c - d$ we obtain $a \geq c$ and $a \geq b$. For example, the matrix $\mathbf{A}$ can be

$$\mathbf{A} = \begin{bmatrix} 10 & 4 \\ 5 & 1 \end{bmatrix}.$$

This matrix has a saddle point in the first row and the second column. Thus, the game will have an equilibrium in pure strategies. We can proceed analogously in the three remaining cases: $b \geq d$ and $c < d$, $b < d$ and $c \geq d$, $b < d$ and $c < d$.

Solution 4.

It suffices to show that the function u has no saddle point. This is equivalent to the fact that the minmax value $\overline{v}$ of Player 2 is strictly greater than maxmin value $\underline{v}$ of Player 1. Specifically, these values are

$$\begin{aligned} \underline{v} &= \max_{s_1 \in X} \underline{f}(s_1), \\ \overline{v} &= \min_{s_2 \in X} \overline{f}(s_2), \end{aligned}$$

where $\underline{f}$ and $\overline{f}$ are the functions defined by

$$\underline{f}(s_1) = \min_{s_2 \in X} u(s_1, s_2),$$
$$\overline{f}(s_2) = \max_{s_1 \in X} u(s_1, s_2),$$

for all $s_1, s_2 \in X$. It is easy to check that

$$\underline{f}(s_1) = \begin{cases} \frac{1}{1+(s_1-1)^2} & 0 \le s_1 \le \frac{1}{2}, \\ \frac{1}{1+s_1^2} & \frac{1}{2} < s_1 \le 1, \end{cases} \qquad \text{for all } s_1 \in X,$$

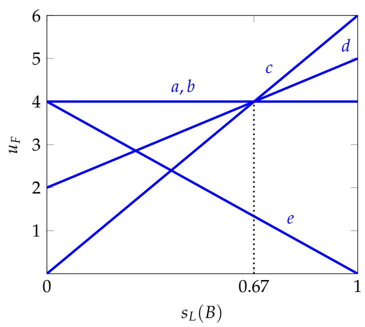
and

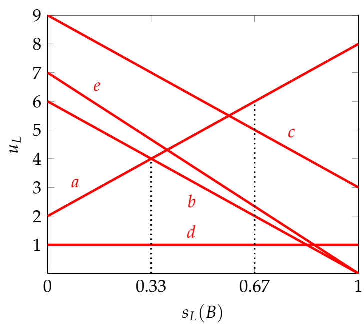
$$\overline{f}(s_2) = 1, \; s_2 \in X.$$

Hence $\underline{v}_1 = \underline{f}(\frac{1}{2}) = \frac{4}{5} < \overline{v}_1 = 1$, so u has no saddle point. Thus, the game has no equilibrium in pure strategies. However, note that it must have at least one equilibrium in mixed strategies by von Neumann's theorem.

Solution 5.

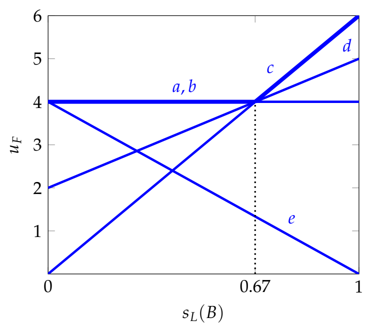
Since the leader has only two actions, we may visualize the expected utility based on the policy announced by the leader.





When computing the Stackelberg equilibrium, the follower always takes a best response to the leader's policy. Best responses are as follows

$$\mathrm{BR}(s_L) = \begin{cases} a, b, e & s_L(B) = 0, \\ a, b & 0 < s_L(B) < \frac{2}{3}, \\ a, b, c, d & s_L(b) = \frac{2}{3}, \\ c & \frac{2}{3} < s_L(b) \le 1, \end{cases}$$

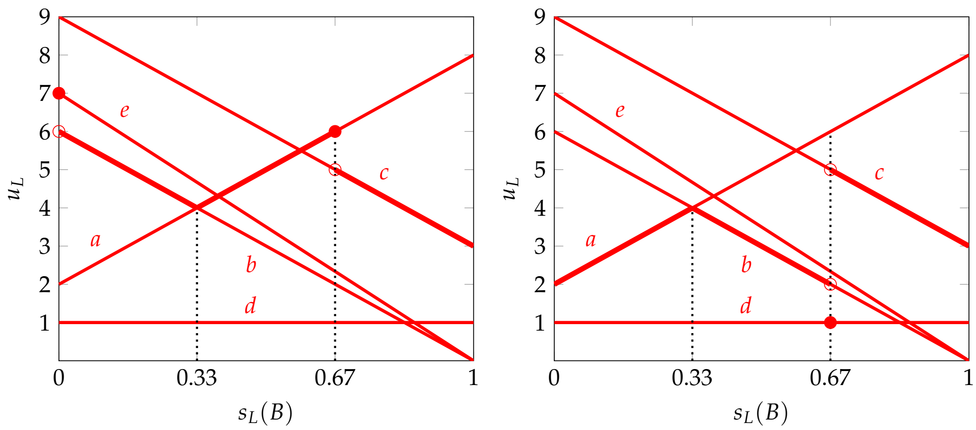


Now we know, based on the policy of the leader, what are the follower best responses. Now we have to figure out which best response the follower would take.
Let us define functions $\overline{\beta}(s_L)$ and $\underline{\beta}(s_L)$, which, based on the leader's policy, give pure action that the follower should play to maximize or minimize the leader's utility. Similarly we define function $\overline{\beta}(s_L)$ and $\underline{\beta}(s_L)$ that corresponds to the expected utility for leader when it fixes its policy.

$$\overline{\beta}(s_L) = \arg\max_{a_F \in BR(s_L)} u_L(s_L, a_F) \qquad \underline{\beta}(s_L) = \arg\min_{a_F \in BR(s_L)} u_L(s_L, a_F)$$

The policies are then

$$\overline{\beta}(s_L) = \begin{cases} e & s_L(B) = 0, \\ b & 0 < s_L(B) \le \frac{1}{3}, \\ a & \frac{1}{3} < s_L(B) \le \frac{2}{3}, \\ c & \frac{2}{3} < s_L(B) \le 1, \end{cases} \qquad \underline{\beta}(s_L) = \begin{cases} a & 0 \le s_L(B) < \frac{1}{3}, \\ b & \frac{1}{3} \le s_L(B) < \frac{2}{3}, \\ d & s_L(B) = \frac{2}{3}, \\ c & \frac{2}{3} < s_L(B) \le 1, \end{cases}$$



Strong Stackelberg equilibrium from the left plot is strategy $s_L(B) = 0$ because this maximizes the leader's utility. Weak Stackelberg equilibrium does not exist, because if we take $\varepsilon > 0$ and policy $s_L(B) = \frac{2}{3} + \varepsilon$. When lowering the value of ε, we always get better utility, and when $\varepsilon = 0$, the value of action c is 5, which is the best we can get. But when $\varepsilon = 0$, the utility drops to 1, because optimal action changes to d. Therefore we cannot set such ε, which would maximize the value, so the Weak Stackelberg equilibrium does not exist.

Solution 6.

Matching Pennies is a two-player zero-sum game, so we can formulate the equilibrium problem as a linear programming problem, analogously to Exercise 1. However, we will take a different approach and find the solution in a more elementary way.

First, it can be easily checked that the game has no solution in pure strategies. Let $x \in [0, 1]$ and $y \in [0, 1]$ denote the probabilities of selecting zero bit for Player 1 and 2, respectively. The expected utility of Player 1 is then the function given by $U(x, y) = 4xy - 2x - 2y + 1$, for every $x, y \in [0, 1]$. For any choice $x \in [0,1]$ of Player 1, Player 2 will select the most harmful strategy for Player 1. This implies that Player 1 gets in this worst case scenario

$$\ell(x) := \min_{y \in [0,1]} U(x,y) = \begin{cases} 2x-1 & 0 \leq x < \frac{1}{2}, \\ 0 & x = \frac{1}{2}, \\ 1-2x & \frac{1}{2} < x \leq 1, \end{cases} \qquad x \in [0,1].$$

Player 1 can secure the maxmin value

$$\max_{x \in [0,1]} \ell(x) \tag{2}$$

which is equal to the minmax value of Player 2 by von Neumann's theorem. The equilibrium strategy of Player 1 is then any solution to the optimization problem (2). The only such solution is $x^* = \frac{1}{2}$. Repeating this analysis for Player 2 and the minmax value, we arrive at the same solution $y^* = \frac{1}{2}$ for Player 2. In conclusion, the only equilibrium strategy for each player is to randomize uniformly between the two choices.