COMPUTATIONAL GAME THEORY

Exercises

TOMÁŠ KROUPA, ONDŘEJ KUBÍČEK, BRANISLAV BOŠANSKÝ*

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1 NORMAL-FORM GAMES

Exercise 1.
Single attacker is about to attack one of the 4 military bases (denoted $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$, $\mathbf{T}_4$). However, due to pay cuts, the military can only spend resources to defend one of these bases. If the attacker attacks the defended bases, it gets a utility of -1, while the military gets 2. On the other hand, when the attack is succesful, the military does not receive anything, but the attacker gets the following rewards based on the base it attacked $\mathbf{T}_1 \to 3$, $\mathbf{T}_2 \to 7$, $\mathbf{T}_3 \to 1$ and $\mathbf{T}_4 \to 5$. (a) Formalize this as a Normal-Form Game. (b) Let us assume that when the attacker attacks base, the alarm is triggered. If attacked base is near the protected one, the military may dispatch striking team, which has 50% chance to interrupt the attack. For base $\mathbf{T}_i$, the nearby bases are $\mathbf{T}_{i-1}$ and $\mathbf{T}_{i+1}$. Bases $\mathbf{T}_1$ and $\mathbf{T}_4$ are also close to each other. (c) Find all pure Nash Equilibria in those games.

Exercise 2.
The two-player normal-form game with the payoff matrix

		Player 2		
		V	**S**	**R**
	U	$1,3$	$4,2$	$-1,2$
Player 1	**C**	$1,0$	$2,-2$	$0,-1$
	D	$1,2$	$-1,1$	$3,3$

(a) Find all Pareto optimal outcomes. (b) Find all pure Nash equilibria. (c) Find all dominated pure strategies and apply iterative removal of these strategies.

* *AI Center, Department of Computer Science, Faculty of Electrical Engineering, Czech Technical University in Prague*

Exercise 3.

The concept of correlated equilibrium is a generalization of Nash equilibrium in the following sense. Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be an n-player strategic game, $(p_1, \ldots, p_n)$ be its Nash equilibrium in mixed strategies, and define

$$p(s_1, \ldots, s_n) := \prod_{i \in N} p_i(s_i), \qquad (s_1, \ldots, s_n) \in S = S_1 \times \cdots \times S_n.$$

Then p is a correlated equilibrium of G. Prove this claim.

Exercise 4.

Alice and Bob play one round of a zero-sum game captured by the payoff matrix of Alice:

		Bob		
		e	f	g
Alice	a	6	0	−1
	b	5	4	9
	c	9	−3	−1
	d	−1	1	−1

Alice reveals publicly that she will be using strategy b. After making her choice public she must stick to it. Can Bob take advantage of knowing the strategic choice of Alice compared to the standard situation when Alice's strategic choice wouldn't be known a priori?

SOLUTIONS

Solution 1.

(a) We use the military as a row player and the attacker as a column player. The first value corresponds to the utility of the defender, while the second corresponds to the attacker's utility. The utility matrix in the Normal-Form Game is then

		Attacker			
		$\mathbf{T}_1$	$\mathbf{T}_2$	$\mathbf{T}_3$	$\mathbf{T}_4$
Military	$\mathbf{T}_1$	$2, -1$	$0, 7$	$0, 1$	$0, 5$
	$\mathbf{T}_2$	$0, 3$	$2, -1$	$0, 1$	$0, 5$
	$\mathbf{T}_3$	$0, 3$	$0, 7$	$2, -1$	$0, 5$
	$\mathbf{T}_4$	$0, 3$	$0, 7$	$0, 1$	$2, -1$

(b) In the changed game, when the attacker attacks defended base, the result is still the same. But when the base is nearby the protected base, the utility changes in the following way.

$$u_1(\mathbf{T}_i, \mathbf{T}_{i+1}) = \frac{1}{2}u_1(\mathbf{T}_{i+1}, \mathbf{T}_{i+1}) + \frac{1}{2} \cdot 2$$
$$u_2(\mathbf{T}_i, \mathbf{T}_{i+1}) = \frac{1}{2}u_2(\mathbf{T}_{i+1}, \mathbf{T}_{i+1}) + \frac{1}{2} \cdot (-1)$$

The utility matrix in changed game is

		Attacker			
		$\mathbf{T}_1$	$\mathbf{T}_2$	$\mathbf{T}_3$	$\mathbf{T}_4$
Military	$\mathbf{T}_1$	$2, -1$	$1, 3$	$0, 1$	$1, 2$
	$\mathbf{T}_2$	$1, 1$	$2, -1$	$1, 0$	$0, 5$
	$\mathbf{T}_3$	$0, 3$	$1, 3$	$2, -1$	$1, 2$
	$\mathbf{T}_4$	$1, 1$	$0, 7$	$1, 0$	$2, -1$

(c) Neither game has any pure Nash Equilibrium.

Solution 2.

(a) Pareto optimal strategy is such a strategy profile in which neither player can improve its utility without decreasing the utility of any other player. Strategy. For example $(\mathbf{U}, \mathbf{V})$ is not Pareto optimal strategy, because when choosing strategy $(\mathbf{D}, \mathbf{R})$, the row player improves its utility from 1 to 3, while columns player utility remains unchanged. The only Pareto optimal strategies are $(\mathbf{D}, \mathbf{R})$ and $(\mathbf{U}, \mathbf{S})$ with utilities

$$u_1(\mathbf{D}, \mathbf{R}) = 3 \quad u_2(\mathbf{D}, \mathbf{R}) = 3$$
$$u_1(\mathbf{U}, \mathbf{S}) = 4 \quad u_2(\mathbf{U}, \mathbf{S}) = 2$$

(b) Nash Equilibrium is a strategy profile where neither player can improve its utility by changing its strategy. For example $(\mathbf{U}, \mathbf{S})$ is not a Nash Equilibrium, because column player may choose action $\mathbf{V}$ and its utility would improve from 2 to 3. Nash equilibria are $(\mathbf{D}, \mathbf{R})$, $(\mathbf{U}, \mathbf{V})$, $(\mathbf{C}, \mathbf{V})$.

(c) Strategy s_i of a player dominates different strategy if, regardless of the opponent's strategy, the utility of s_i is always greater than that of s_j. Similarly, for weak domination, the utility of s_i is always greater or equal than the utility of s_j, and it is strictly greater for at least one opponent's strategy. Pure strategy $\mathbf{V}$ strictly dominates $\mathbf{S}$ (also $\mathbf{R}$ weakly dominates $\mathbf{S}$).

Player 2

		V	R
Player 1	U	1, 3	−1, 2
	C	1, 0	0, −1
	D	1, 2	3, 3

After removing **S**, both **U** and **C** are weakly dominated by **D**.

Player 2

		V	R
Player 1	D	1, 2	3, 3

Finally now **R** strictly dominates **V**. This leaves only strategy profile $(\mathbf{D}, \mathbf{R})$, which is also a Nash equilibrium.

Solution 3.

Clearly, p is a probability distribution on the joint strategy space S by the definition. We know that p is a correlated equilibrium if the inequality

$$\sum_{s_{-i} \in S_{-i}} p(s_i, s_{-i}) \cdot u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} p(s_i, s_{-i}) \cdot u_i(s_i', s_{-i}). \quad (1)$$

holds for each player i and all strategies $s_i, s_i' \in S_i$. Since $(p_1, \ldots, p_n)$ is a Nash equilibrium, it follows that for each player $i \in N$ and all strategies $s_i, s_i' \in S_i$ such that $p_i(s_i) > 0$,

$$U_i(s_i, p_{-i}) \geq U_i(s_i', p_{-i}), \quad (2)$$

where U_i denotes the expected utility (payoff) of player i. Since

$$p_i(s_i) \cdot U_i(s_i, p_{-i}) = \sum_{s_{-i} \in S_{-i}} p_i(s_i) \cdot \prod_{j \neq i} p_j(s_j) \cdot u_i(s_i, s_{-i}) = \sum_{s_{-i} \in S_{-i}} p(s_i, s_{-i}) \cdot u_i(s_i, s_{-i})$$

and

$$p_i(s_i) \cdot U_i(s_i', p_{-i}) = \sum_{s_{-i} \in S_{-i}} p_i(s_i) \cdot \prod_{j \neq i} p_j(s_j) \cdot u_i(s_i', s_{-i}) = \sum_{s_{-i} \in S_{-i}} p(s_i, s_{-i}) \cdot u_i(s_i', s_{-i}),$$

from (2) we get (1). In case that $p_i(s_i) = 0$, necessarily $p(s_i, s_{-i}) = 0$ for every $s_{-i} \in S_{-i}$, so that boths sides of (1) are zero.

Solution 4.

First, note that the analyzed game has a unique pure strategy equilibrium (b, f) and the value of game is equal to 4:

Bob

		e	**f**	*g*
Alice	*a*	6	0	−1
	b	5	4	9
	c	9	−3	−1
	d	−1	1	−1

Knowing that the choice of Alice is b, her equilibrium strategy, Bob plays strategy f since this is his best response strategy. However, note that revealing publicly the equilibrium strategy doesn't help the opposite player in any zero-sum game. Indeed, if Bob doesn't know the strategy of Alice, he would play the strategy that guarantees him the minimal loss for every possible strategic choice of Alice, which is precisely the minmax strategy f.