

Computational Game Theory

The Nucleolus

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How to divide the estate among claimants?

- After the death of a man, 3 creditors raise claims
- We consider 3 different amounts of estate

Allocations according to the Talmud for different estates

Aumann and Maschler (1985)

Estate/Demand	100	200	300
100	$100/3$	$100/3$	$100/3$
200	50	75	75
300	50	100	150

From bankruptcy problems to bankruptcy games

Let $N = \{1, \dots, n\}$ be the set of claimants.

Definition

A **bankruptcy problem** is a pair (s, \mathbf{d}) , where $s \geq 0$ is the estate and $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{R}_+^n$ are the demands such that

$$s \leq d_1 + \dots + d_n.$$

Definition

A **bankruptcy game** associated with a bankruptcy problem (s, \mathbf{d}) is a coalitional game given by

$$v(A) = \max \{s - \mathbf{d}(N \setminus A), 0\}, \quad A \subseteq N.$$

Solving bankruptcy games

Every bankruptcy game is **supermodular**, which implies that

- The core $\mathcal{C}(v)$ is nonempty and
- The Shapley value belongs to $\mathcal{C}(v)$

Example based on Table 1

$$s = 200, \mathbf{d} = (100, 200, 300), \text{ and } v(A) = \begin{cases} 200 & A = N, \\ 100 & A = 23, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{C}(v) = \text{conv}\{(100, 100, 0), (100, 0, 100), (0, 200, 0), (0, 0, 200)\}$$

$$\varphi^S(v) = \frac{1}{3} \cdot (100, 250, 250)$$

We will study a division rule different from the Shapley value

- It applies to all coalitional games
- It coincides with the Talmud rule for bankruptcy problems
- it can be considered as a refinement of the core solution

The idea is that the maximal dissatisfaction of coalitions with an allocation should be minimized

The nucleolus

Measuring excess of coalitions in game v

Let $\mathbf{x} \in \mathbb{R}^n$ be an allocation. The **excess** of coalition $A \subseteq N$ at \mathbf{x} is

$$e(A, \mathbf{x}) = v(A) - \mathbf{x}(A).$$

Definition

Enumerate coalitions from the highest excess:

$$e(A_1, \mathbf{x}) \geq \dots \geq e(A_{2^n}, \mathbf{x}).$$

The **excess vector** at \mathbf{x} is

$$e(\mathbf{x}) = (e(A_1, \mathbf{x}), \dots, e(A_{2^n}, \mathbf{x})) \in \mathbb{R}^{2^n}.$$

Which allocations should be considered for measuring excess?

- The core is difficult to compute or even empty
- We relax the core constraints mildly...
- ...and define the set of **imputations**:

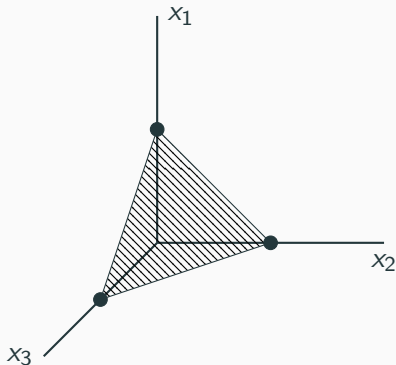
$$\mathcal{I}(v) = \{ \mathbf{x} \in \mathbb{R}^n \mid \underbrace{\mathbf{x}(N) = v(N)}_{\text{Efficiency}}, \underbrace{x_i \geq v(i), \forall i \in N}_{\text{Individual rationality}} \}$$

Claim

If v is a superadditive game, then $\mathcal{I}(v) \neq \emptyset$

Example: Imputation set for games with $N = \{1, 2, 3\}$

$$v(N) > 0, \quad v(1) = v(2) = v(3) = 0$$



$$\mathcal{I}(v) = \{ \mathbf{x} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = v(N), \quad x_1, x_2, x_3 \geq 0 \}$$

Minimizing the excess mapping

1. The players consider the set of imputations $\mathcal{I}(v)$ in game v
2. This set is mapped to the excess vectors in \mathbb{R}^{2^n} ,

$$e: \mathbf{x} \in \mathcal{I}(v) \rightarrow e(\mathbf{x}) \in \mathbb{R}^{2^n}$$

3. We want to find a **minimizer** of e over $\mathcal{I}(v)$

However, the objective e is vector-valued!

Definition

For every $\alpha, \beta \in \mathbb{R}^m$, define:

- $\alpha \prec \beta$ if there is $k = 1, \dots, m$ such that for each $j < k$, $\alpha_j = \beta_j$ and $\alpha_k < \beta_k$
- $\alpha \preceq \beta$ if $\alpha \prec \beta$ or $\alpha = \beta$

The binary relation \preceq is a *total order* on \mathbb{R}^m .

The nucleolus

Definition

Let v be a game with $\mathcal{I}(v) \neq \emptyset$. The **nucleolus** of v is the set

$$\mathcal{N}(v) = \{\mathbf{x} \in \mathcal{I}(v) \mid e(\mathbf{x}) \preceq e(\mathbf{y}) \text{ for all } \mathbf{y} \in \mathcal{I}(v)\}$$

1. Is $\mathcal{N}(v)$ nonempty?
2. Is $\mathcal{N}(v)$ single-valued?
3. How to compute $\mathcal{N}(v)$?

Existence of the nucleolus

Theorem (Schmeidler, 1969)

Let v be a game with $\mathcal{I}(v) \neq \emptyset$. Then $|\mathcal{N}(v)| = 1$.

Properties of the nucleolus

- If $\mathcal{C}(v) \neq \emptyset$, then $\mathcal{N}(v) \in \mathcal{C}(v)$
- Efficiency
- Symmetry
- Null player property

Example

Glove game

$$N = \{1, 2, 3\} \quad v(A) = \begin{cases} 1 & A = 12, 13, N, \\ 0 & \text{otherwise.} \end{cases}$$

Some imputations: $\mathbf{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $\mathbf{y} = (1, 0, 0)$, $\mathbf{z} = (\frac{4}{6}, \frac{1}{6}, \frac{1}{6})$

A	\emptyset	1	2	3	12	13	23	N
$e(A, \mathbf{x})$	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	0
$e(A, \mathbf{y})$	0	-1	0	0	0	0	0	0
$e(A, \mathbf{z})$	0	$-\frac{2}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$	0

$$e(\mathbf{y}) \prec e(\mathbf{z}) \prec e(\mathbf{x})$$

Since \mathbf{y} is the only core element, it is also the nucleolus.

Example: Solution to the original bankruptcy problem

Example based on Table 1

$$s = 200, \mathbf{d} = (100, 200, 300), \text{ and } v(A) = \begin{cases} 200 & A = N, \\ 100 & A = 23, \\ 0 & \text{otherwise.} \end{cases}$$

Consider $\mathbf{x} = (50, 75, 75)$ and any $\mathbf{y} \in \mathcal{C}(v)$ to show $e(\mathbf{x}) \preceq e(\mathbf{y})$:

A	1	2	3	12	13	23
$e(A, \mathbf{x})$	-50	-75	-75	-125	-125	-50
$e(A, \mathbf{y})$	$-y_1$	$-y_2$	$y_1 + y_2 - 200$	$-y_1 - y_2$	$y_2 - 200$	$y_1 - 100$

The nucleolus of a two-player game

Example

Consider a superadditive game v with two players:

$$v(12) \geq v(1) + v(2)$$

- The set of imputations is the line segment

$$\mathcal{I}(v) = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 + x_2 = v(12), x_1 \geq v(1), x_2 \geq v(2)\}$$

- The nucleolus is the allocation

$$\left(v(1) + \frac{v(12) - v(1) - v(2)}{2}, v(2) + \frac{v(12) - v(1) - v(2)}{2} \right)$$

How to compute the nucleolus?

Algorithm

Input: Game v such that $\mathcal{I}(v) \neq \emptyset$

1. Find $X_1 \subseteq \mathcal{I}(v)$ minimizing the maximal excess
2. Find $X_2 \subseteq X_1$ minimizing the second highest excess
3. Continue this procedure...
4. ... until it yields a single imputation, the nucleolus

Minimizing the maximal excess

LP with variables $\mathbf{x} = (x_1, \dots, x_n), t$

Minimize t
subject to $e(A, \mathbf{x}) \leq t, \quad \emptyset \neq A \subset N,$
 $\mathbf{x} \in \mathcal{I}(v)$

t_1 = the value of the LP

$X_1 \times \{t_1\}$ = the set of optimal solutions

- If X_1 is a singleton, then $X_1 = \mathcal{N}(v)$
- Else put

$$\mathcal{F}_1 = \{A \subset N \mid e(A, \mathbf{x}) = t_1, \mathbf{x} \in X_1\}$$

Minimizing the second highest excess

LP with variables $\mathbf{x} = (x_1, \dots, x_n), t$

Minimize t
subject to $e(A, \mathbf{x}) \leq t, \quad A \notin \mathcal{F}_1, \quad \emptyset \neq A \subset N$
 $\mathbf{x} \in X_1$

t_2 = the value of the LP

$X_2 \times \{t_2\}$ = the set of optimal solutions

- If X_2 is a singleton, then $X_2 = \mathcal{N}(v)$
- Else put

$$\mathcal{F}_2 = \{A \subset N \mid e(A, \mathbf{x}) = t_2, \mathbf{x} \in X_2\}$$

Minimizing the k -th highest excess

The algorithm stops when X_k is a singleton at step $k \leq 2^n$.

- Each t_i is the i -th highest excess
- Each \mathcal{F}_i is the collection of coalitions with excess t_i
- At each step, \mathcal{F}_i contains at least one new coalition

Computing the nucleolus

- NP-hard for many classes of games (weighted voting games)
- For some games efficient algorithms exist:



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Summary: Properties of solution concepts

Property/Solution	<i>core</i>	<i>Shapley value</i>	<i>Banzhaf value</i>	<i>nucleolus</i>
Nonemptiness	—	✓	✓	☞
Efficiency	✓	✓	—	✓
Individual rationality	✓	☞	☞	✓
Symmetry	—	✓	✓	✓
Null player property	✓	✓	✓	✓
Additivity	—	✓	✓	—

☞ This property is true for every superadditive game



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