

# Computational Game Theory

## Coalitional Games and the Core

---

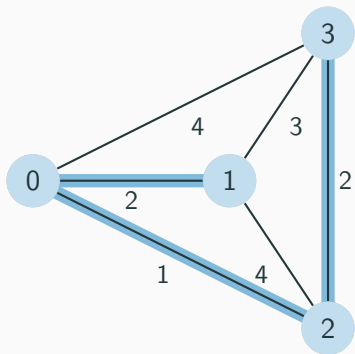
Tomáš Kroupa

2022

Department of Computer Science  
Faculty of Electrical Engineering  
Czech Technical University in Prague

## How to share the cost?

- Cities 1, 2, 3 need to connect to the *provider* of energy 0
- The graph shows costs of pairwise connections
- For each  $A \subseteq \{1, 2, 3\}$ , the cost of connecting  $A$  to 0 is determined by a **minimum cost spanning tree** on  $A \cup \{0\}$



$$c(A) = \begin{cases} 2 & A = 1 \\ 1 & A = 2 \\ 4 & A = 3 \\ 3 & A = 12, 23 \\ 5 & A = 13, 123 \end{cases}$$

# How to determine the voting power?

## UN Security Council

- 5 *permanent* and 10 *non-permanent* members
- A resolution is approved by all the permanent members and at least 4 non-permanent members
- There are  $2^{15}$  voting scenarios

# What is it all about?

- **Fair** division of costs
- **Power** of agents controlling some resources
- **Fairness** of a complicated voting system
- **Efficient** allocation of the profit among agents
- **Influence** of features in a predictive ML model

# Game forms

1. Normal (Strategic)
2. Extensive
3. Coalitional

## Games in coalitional form

- Players can form *coalitions*
- A coalition is a set of players coordinating their strategies in order to maximize *the utility of the coalition*
- Strategic aspects of coalitional games are unimportant, since they are implicitly part of the deal among players

# Players and coalitions

- The **player set** is

$$N = \{1, \dots, n\}, \quad \text{for some } n \in \mathbb{N}$$

- A **coalition** is a subset  $A \subseteq N$ , where
  - $\emptyset$  is the empty coalition
  - $N$  is the *grand coalition*
  - $\{i\}$  is a one-player coalition
- The set of all coalitions is the **powerset**

$$\mathcal{P}(N) = \{A \mid A \subseteq N\}$$

# Coalitional games

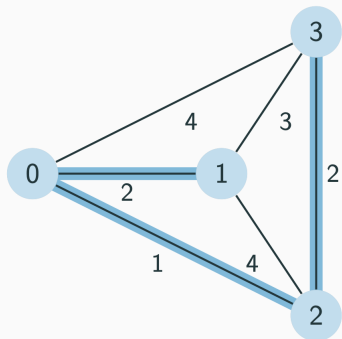
## Definition

**Coalitional game** is a pair  $(N, v)$ , where  $v$  is a function

$$v: \mathcal{P}(N) \rightarrow \mathbb{R} \quad \text{such that } v(\emptyset) = 0.$$

- The players in coalition  $A$  receive the **worth**  $v(A)$  independently of the actions of players in  $N \setminus A$
- We will identify a coalitional game  $(N, v)$  with function  $v$  when the player set  $N$  is understood

## Example: Savings game $v$



$c(A)$  = cost of connecting  $A$

$$v(A) = \sum_{i \in A} c(i) - c(A)$$

$$= \begin{cases} 0 & A = \emptyset, 1, 2, 3, 12 \\ 1 & A = 13 \\ 2 & A = 23, 123 \end{cases}$$

Game  $v$  is **superadditive**:

$$v(A) + v(B) \leq v(A \cup B) \quad \text{if } A \cap B = \emptyset.$$



## Example: Voting game

### UN Security Council

- 5 *permanent* and 10 *non-permanent* members
- A resolution is approved by all the permanent members and at least 4 non-permanent members

$$v(A) = \begin{cases} 1 & \text{if } A \supseteq \{1, \dots, 5\} \text{ and } |A| \geq 9, \\ 0 & \text{otherwise.} \end{cases}$$

Game  $v$  is superadditive. Moreover,  $v$  is **simple**:

- $v(A) \in \{0, 1\}$  and  $v(N) = 1$ ,
- $v$  is **monotone**:  $v(A) \leq v(B)$  for all  $A \subseteq B$ .

# Main questions of coalitional game theory

1. **Which** coalitions will form?
2. **How** a coalition allocates its worth to its members?

## Which coalitions will form?

- A **coalitional structure** is a partition  $\mathcal{S} = \{A_1, \dots, A_k\}$  of  $N$ :
  1.  $A_1 \cup \dots \cup A_k = N$ , where  $A_i \neq \emptyset$
  2.  $A_i \cap A_j = \emptyset$  for all  $i \neq j$
- The total utility of  $\mathcal{S}$  is then

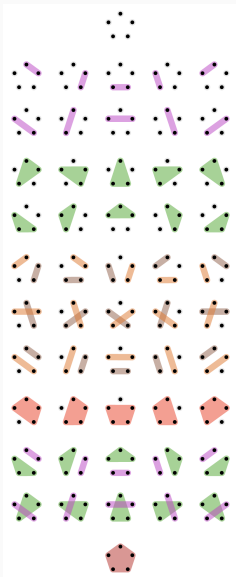
$$V(\mathcal{S}) = \sum_{i=1}^k v(A_i)$$

### Coalition formation problem

Find a coalitional structure  $\mathcal{S}^*$  satisfying

$$V(\mathcal{S}^*) = \max \{V(\mathcal{S}) \mid \mathcal{S} \text{ is a coalitional structure}\}$$

## Example: Coalitional structures for five players



Source: Wikipedia

# Coalition formation problem

- **Bell numbers**  $B_n$  count the number of coalitional structures:

$n$	3	...	10	...	15
$B_n$	15	...	115 975	...	1 382 958 545

- Finding an optimal coalitional structure  $\mathcal{S}^*$  is NP-complete

## Trivial solution for superadditive games

Let  $v$  be a superadditive game. For any coalitional structure  $\mathcal{S}$ ,

$$V(\mathcal{S}) = \sum_{i=1}^k v(A_i) \leq v(N) = V(\{N\}).$$

This implies that  $\mathcal{S}^* = \{N\}$ .

# Main questions revisited

## Which coalitions will form?

- We assume that players form grand coalition  $N$
- This is optimal for superadditive games

## How a coalition allocates its worth to its members?

- An **allocation** is a vector  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$
- If  $\mathbf{x}$  is allocated to players, coalition  $A \subseteq N$  obtains

$$\mathbf{x}(A) = \sum_{i \in A} x_i$$

*The solution for a game  $v$  is a set of allocations  $\mathbf{x} \in \mathbb{R}^n$ .*

We will study three solution concepts in this course:

1. Core
2. Shapley value
3. Nucleolus

**Core**

---



# What is the core of a game?

*The core is a set of efficient allocations upon which no coalition can improve.*

## Definition

The **core** of a game  $v$  is the set

$$\mathcal{C}(v) = \{ \mathbf{x} \in \mathbb{R}^n \mid \underbrace{\mathbf{x}(N) = v(N)}_{\text{Efficiency}}, \underbrace{\mathbf{x}(A) \geq v(A), \forall A \subseteq N}_{\text{Coalitional rationality}} \}.$$

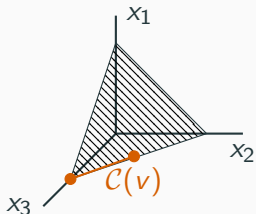
- The core is a convex polytope in  $\mathbb{R}^n$  of dimension  $\leq n - 1$
- Is the core always nonempty? How to find core allocations?

## Example: Savings game $v$

What is the distribution of total saving?

$$v(A) = \begin{cases} 0 & A = \emptyset, 1, 2, 3, 12 \\ 1 & A = 13 \\ 2 & A = 23, 123 \end{cases}$$

$$\begin{aligned} C(v) &= \{ \mathbf{x} \in \mathbb{R}_+^3 \mid x_1 + x_3 \geq 1, x_2 + x_3 \geq 2, x_1 + x_2 + x_3 = 2 \} \\ &= \text{conv} \{ (0, 0, 2), (0, 1, 1) \} \end{aligned}$$

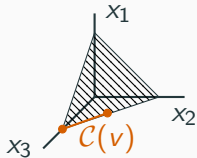
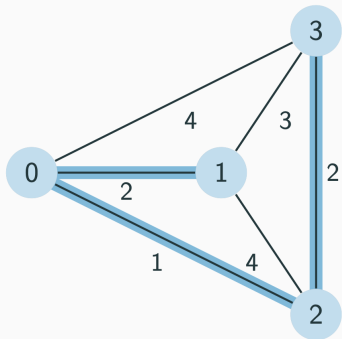


*How to divide the cost?*



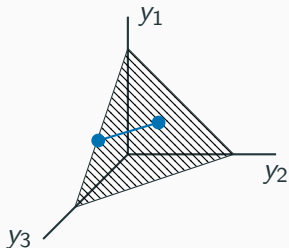
$$y_i = c(i) - x_i$$

## How to divide the cost – a solution



$$y_i = c(i) - x_i$$

$$\mathbf{y} \in \text{conv}\{(2, 0, 3), (2, 1, 2)\}$$



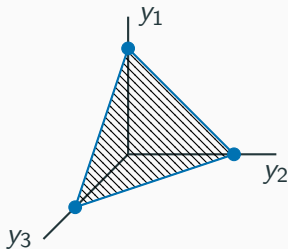
# Games can have empty cores

## Simple majority voting

Three players vote by majority. This determines a game

$$v(A) = \begin{cases} 1 & |A| \geq 2, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for all } A \subseteq \{1, 2, 3\}.$$

Then  $\mathcal{C}(v) = \emptyset$ .



## How to decide nonemptiness of the core

$$\mathcal{C}(v) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}(N) = v(N), \quad \mathbf{x}(A) \geq v(A), \quad \forall A \subseteq N\}$$

**Linear program with real variables**  $x_1, \dots, x_n$

Minimize  $x_1 + \dots + x_n$

subject to  $\sum_{i \in A} x_i \geq v(A)$  for each nonempty  $A \subseteq N$

The following are equivalent:

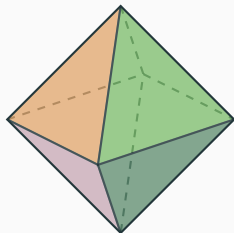
- The optimal value is  $v(N)$
- $\mathcal{C}(v) \neq \emptyset$

# How to find all core allocations

The core  $\mathcal{C}(v)$  has a representation

$$\mathcal{C}(v) = \text{conv}\{\mathbf{x}_1, \dots, \mathbf{x}_k\},$$

where  $\mathbf{x}_1, \dots, \mathbf{x}_k$  are the **vertices** of  $\mathcal{C}(v)$ .



## Vertex enumeration problem

- Find *all* vertices of the core  $\mathcal{C}(v)$
- A hard problem studied in polyhedral geometry

*This problem has a closed-form solution for some games.*

# Games with incentives to join large coalitions

A game  $v$  is **supermodular** if

$$v(A) + v(B) \leq v(A \cup B) + v(A \cap B) \quad \text{for all } A, B \subseteq N.$$

## Proposition

The following are equivalent.

- Game  $v$  is supermodular.
- For all  $A, B \subseteq N$  with  $A \subseteq B$ , and each  $i \in N \setminus B$ ,

$$v(A \cup i) - v(A) \leq v(B \cup i) - v(B).$$

# It is about marginal contributions of players

- Given a **permutation**  $\pi$  of  $N$ , the rank of player  $i$  is  $\pi(i)$
- The coalition preceding player  $i$  is then

$$A_i^\pi = \{j \in N \mid \pi(j) < \pi(i)\}.$$

## Definition

A **marginal vector** is an allocation  $\mathbf{x}^\pi \in \mathbb{R}^n$  such that

$$x_i^\pi = v(A_i^\pi \cup i) - v(A_i^\pi), \quad i \in N.$$



## Example: Marginal vectors in a supermodular game

This three-player game is supermodular:  $v(A) = \begin{cases} 0 & |A| \leq 1, \\ 1 & |A| = 2, \\ 3 & |A| = 3. \end{cases}$

Permutation	Marginal vector
123	(0, 1, 2)
132	(0, 2, 1)
213	(1, 0, 2)
231	(2, 0, 1)
312	(1, 2, 0)
321	(2, 1, 0)

*Observe that each marginal vector is a core allocation.*

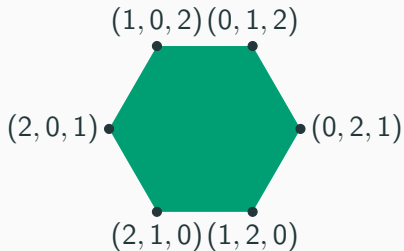
# Cores of supermodular games

## Theorem

The following are equivalent.

- Game  $v$  is supermodular
- The vertices of  $\mathcal{C}(v)$  are precisely marginal vectors

$$v(A) = \begin{cases} 0 & |A| = 1, \\ 1 & |A| = 2, \\ 3 & |A| = 3. \end{cases}$$



# Summing up the core properties

## Pros

- Simple definition
- Core allocations are stable
- Known for some games

## Cons

- May be empty
- May be large
- Hard to compute

*We can seek solution concepts based on different criteria:*

- Nonemptiness
- Single allocation
- **Fairness**