

**DCGI**

**KATEDRA POČÍTAČOVÉ GRAFIKY A INTERAKCE**

# **DUALITY AND APPLICATIONS OF ARRANGEMENTS**

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<https://cw.felk.cvut.cz/doku.php/courses/a4m39vg/start>

Based on [Berg], [Mount], and [Goswami]

Version from 19.12.2019

# Talk overview

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- Duality
  1. Points and lines
  2. Line segments
  3. Polar duality (different points and lines)
  4. Convex hull using duality
- Applications of duality and arrangements

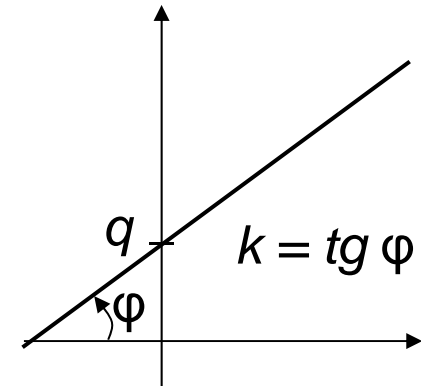


# 1. Duality of lines and points in the plane

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- Points and lines - both have 2 parameters:

- Points – coords  $x$  and  $y$
- Lines – slope  $k$  and  $y$ -intercept  $q$   
 $y = kx + q$



- We can simply **map** points and lines 1:1
- Many mappings exist – it depends on the context



# Why to use duality?

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Some reasons why to use duality:

- Transforming a problem to dual plane may give a **new view on the problem**
- Looking from a different angle may **give the insight** needed to solve it
- Solution in dual space may be even simpler

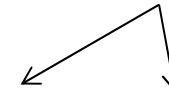


# Definition of duality transformation $D$

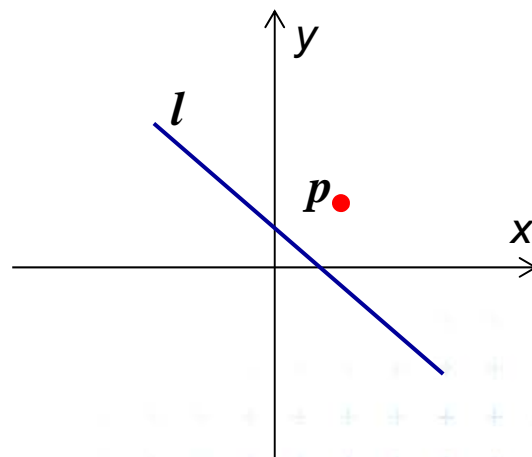
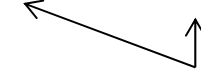
Let  $D$  be the duality transform:

- Point  $p = [p_x, p_y]$  is transformed to line  $D_p = p^* := (b = p_x a - p_y)$
- Line  $l : (y = ax - b)$  is transformed to point  $D_l = l^* := [a, b]$

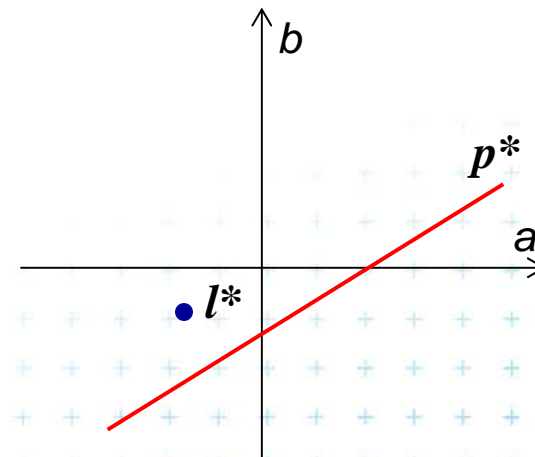
variables



constants



Primal plane ( $xy$ )



Dual plane ( $ab$ )



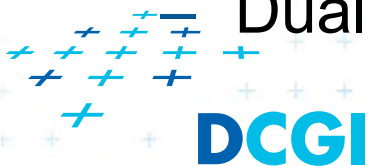
# Example and more about duality $D$

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- Example:  
line  $y = 5x - 3$   
can be represented as point  $y^* = [5, 3]$

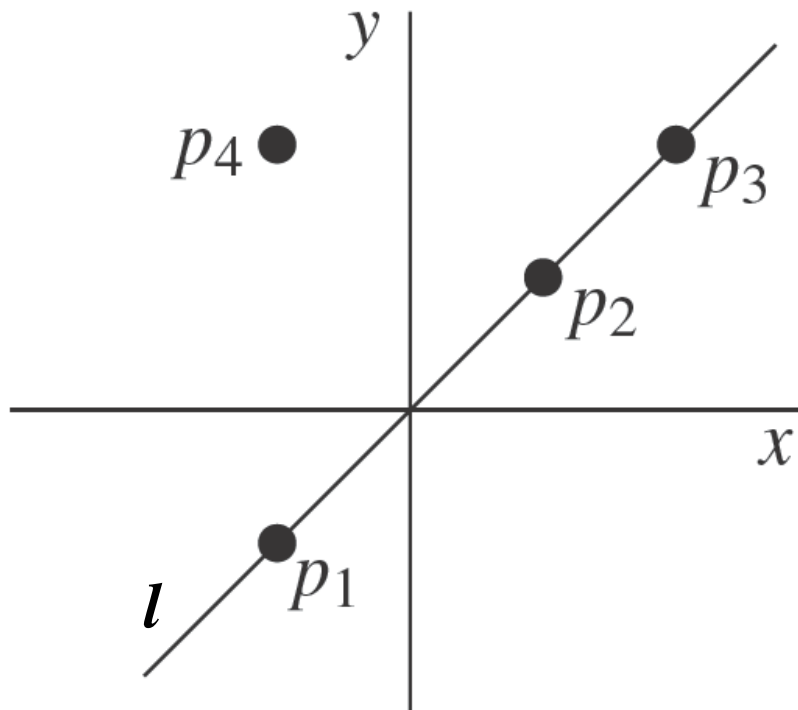
See the [applet]

- Duality  $D$ 
  - is its own **inverse**  $DD_p = p, DD_l = l$
  - cannot represent **vertical lines**  
=> Take vertical lines as special cases, use lexicographic order, or rotate the problem space slightly.
  - Primal plane      – plane with coordinates  $x, y$
  - Dual plane\*      – plane with coordinates  $a, b$

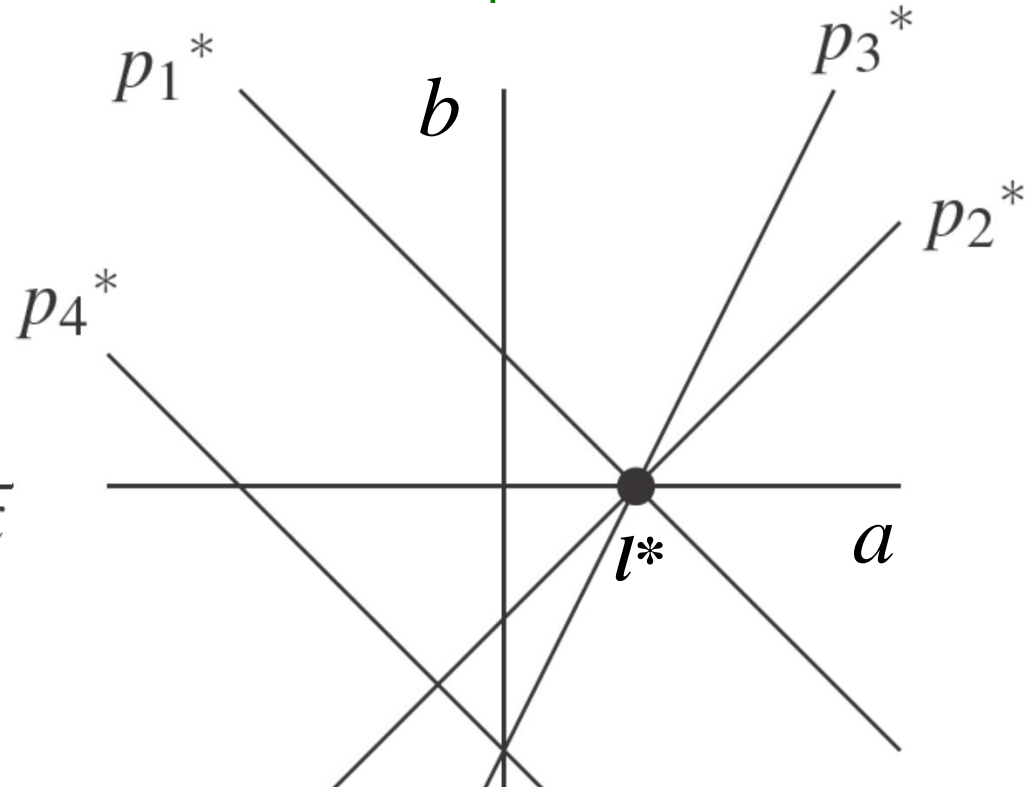


# Duality of lines and points in the plane

Primal plane



Dual plane



point  $p = [p_x, p_y]$

line  $l := (y = ax + b)$

line  $l := (y = ax - b)$

line  $p^* := (b = p_x a - p_y)$

Point  $l^* = [a, -b]$

Point  $l^* = [a, b]$

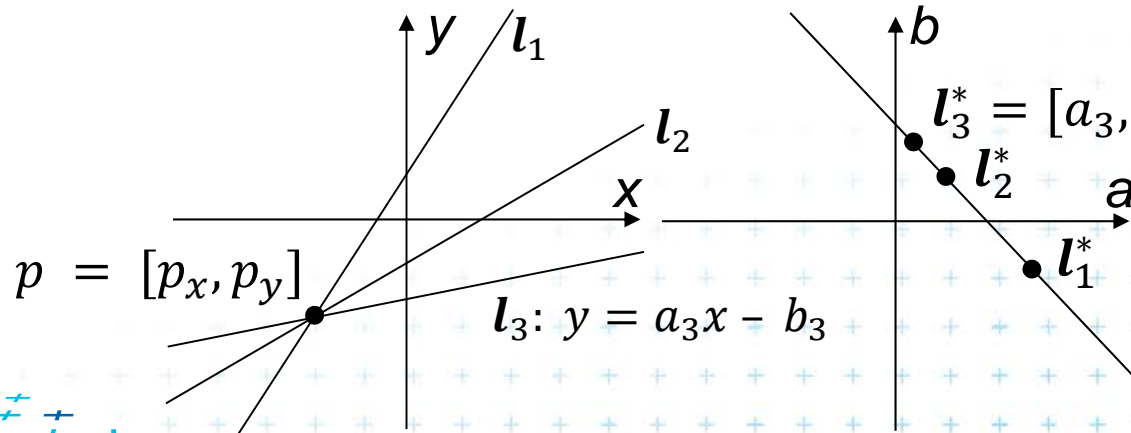
[Berg]

Same form => **It is convenient** to negate  $b$  in the line equation



# Why is $b$ negated in the line equation?

- In primal plane, consider
  - point  $p = [p_x, p_y]$  and
  - set of non-vertical lines  $l_i : y = a_i x - b_i$  passing through  $p$  satisfy the equation  $p_y = a_i p_x - b_i$  (each line with different constants  $a_i, b_i$ )
- In dual plane, these lines transform to collinear points
 
$$\{ l_i^* = [a_i, b_i] : b_i = p_x a_i - p_y \}$$



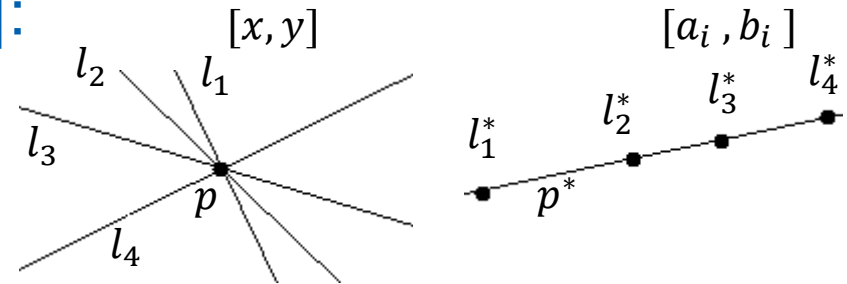
Same form =>  
 It is convenient to negate  $b$  in the line equation





# If $b$ not negated in the line equation...

Lines  $l_i$  have equation  $l_i : y = a_i x - b_i$  OR  $y = a_i x + b_i$  ?  
 passing through point  $p = [p_x, p_y]$ :



- With minus

- equation  $l_i : p_y = a_i p_x - b_i$
- dual points  $\{l_i^* = [a_i, b_i] : b_i = p_x a_i - p_y\}$  ... same form

- With plus

- equation  $l_i : p_y = a_i p_x + b_i$
- dual  $\{l_i^* = [a_i, b_i] : b_i = -p_x a_i + p_y\}$  ... different form

different



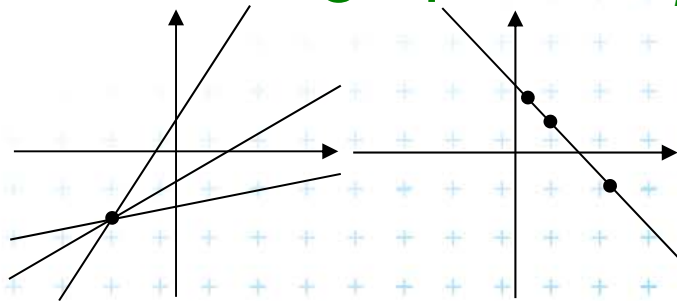
# Properties of points and lines duality

## Incidence is preserved

- Point  $p$  is incident to the line  $l$  in primal plane  
iff  
point  $l^*$  is incident to the line  $p^*$  in the dual plane.



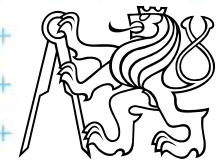
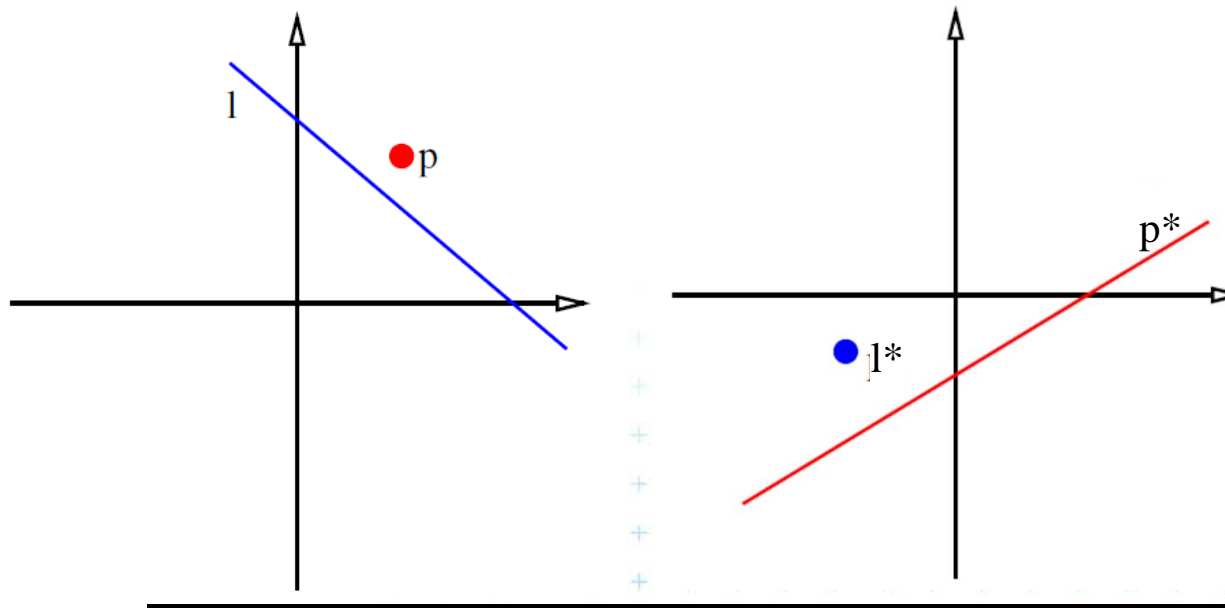
- Lines  $l_1, l_2$  intersect at point  $p$   
iff  
line  $p^*$  passes through points  $l_1^*, l_2^*$ .



# Properties of points and lines duality

But **order is reversed**

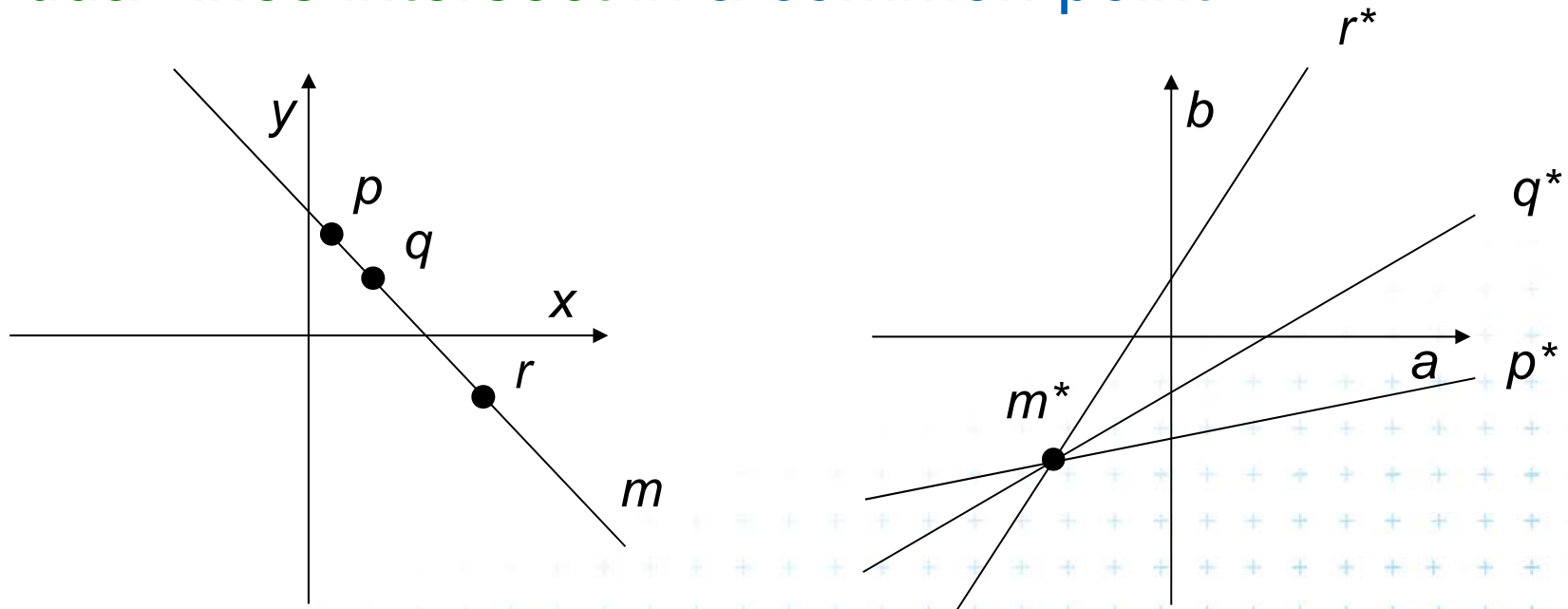
- Point  $p$  lies **above (below)** line  $l$  in the primal plane **iff** line  $p^*$  passes **below (above)** point  $l^*$  in the dual plane Or said order is preserved: ... **iff** Point  $l^*$  lies **above (below)** line  $p^*$



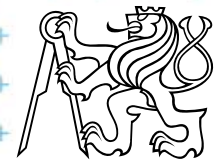
# Properties of points and lines duality

## Collinearity

- Points are **collinear** in the primal plane **iff** their dual lines intersect in a common point

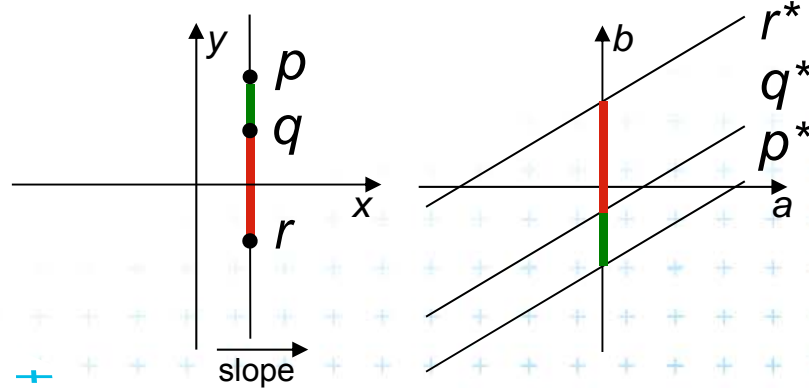


- This does not hold for points on vertical line



# Handling of vertical lines

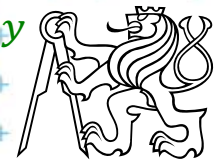
- Dual transform is undefined for vertical lines
  - Points with the **same x** coordinate dualize to lines with the **same slope** (parallel lines) and therefore
  - These dual lines do not intersect (as should for collinear points)
  - **Vertical line** through these points **does not dualize to an intersection point** (would be in infinity, as  $(-\infty, -p_y) \downarrow$  or  $(+\infty, -p_y) \uparrow$ )
  - For detection of vertically collinear points use other **method** -  $O(n)$  vertical lines  $\rightarrow O(n^2)$  brute force 3|| lines s.



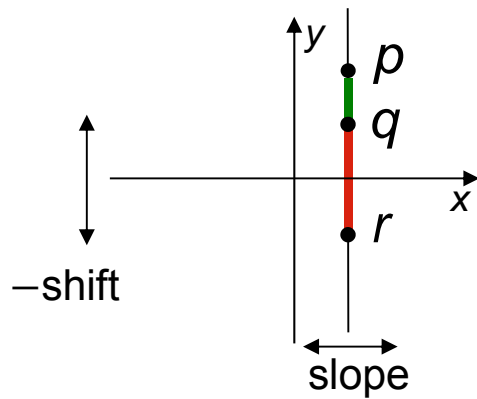
$\rightarrow O(n)$  after  $O(n \log n)$  sorting by  $x$  in primal plane

Vertical distances of such duals are "preserved". For  $p_x = q_x$

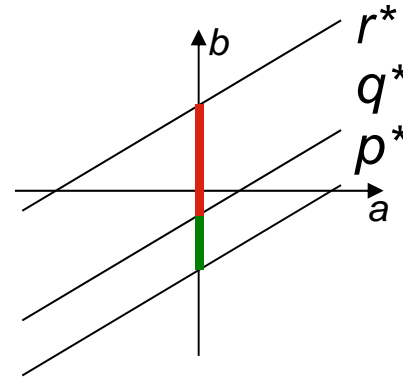
$$\text{vertDist}(q_b^*, p_b^*) = p_y - q_y$$



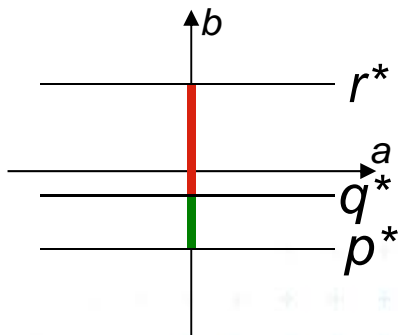
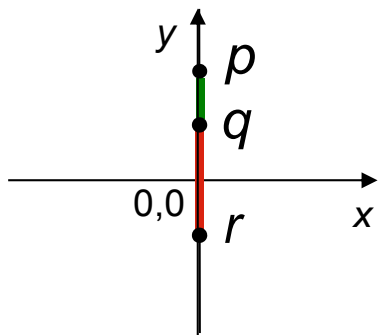
# Vertically collinear points



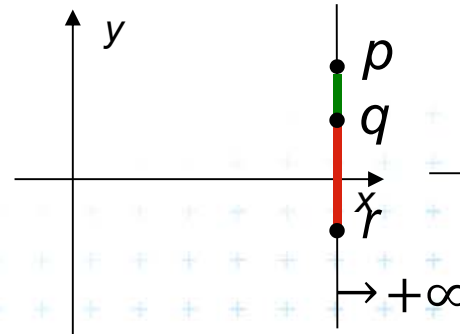
Vertical points



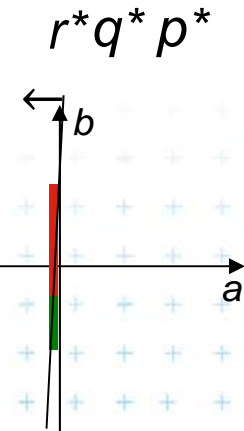
Parallel lines



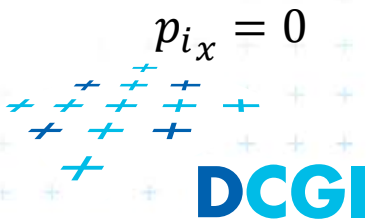
Horizontal lines



$p_{i_x} \rightarrow \infty$



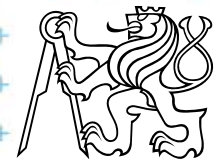
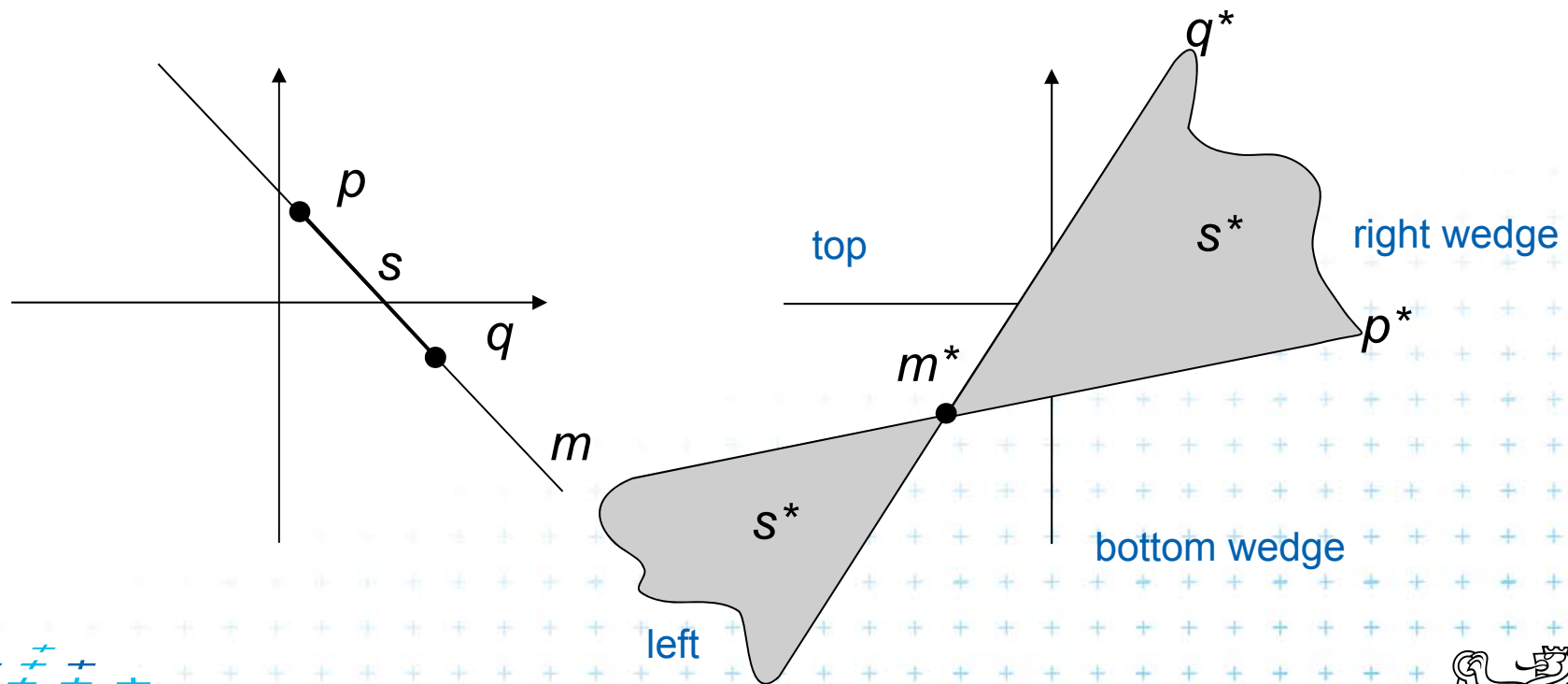
Vertical lines



## 2. Duality of line segments

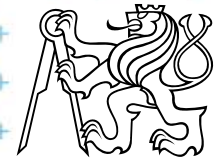
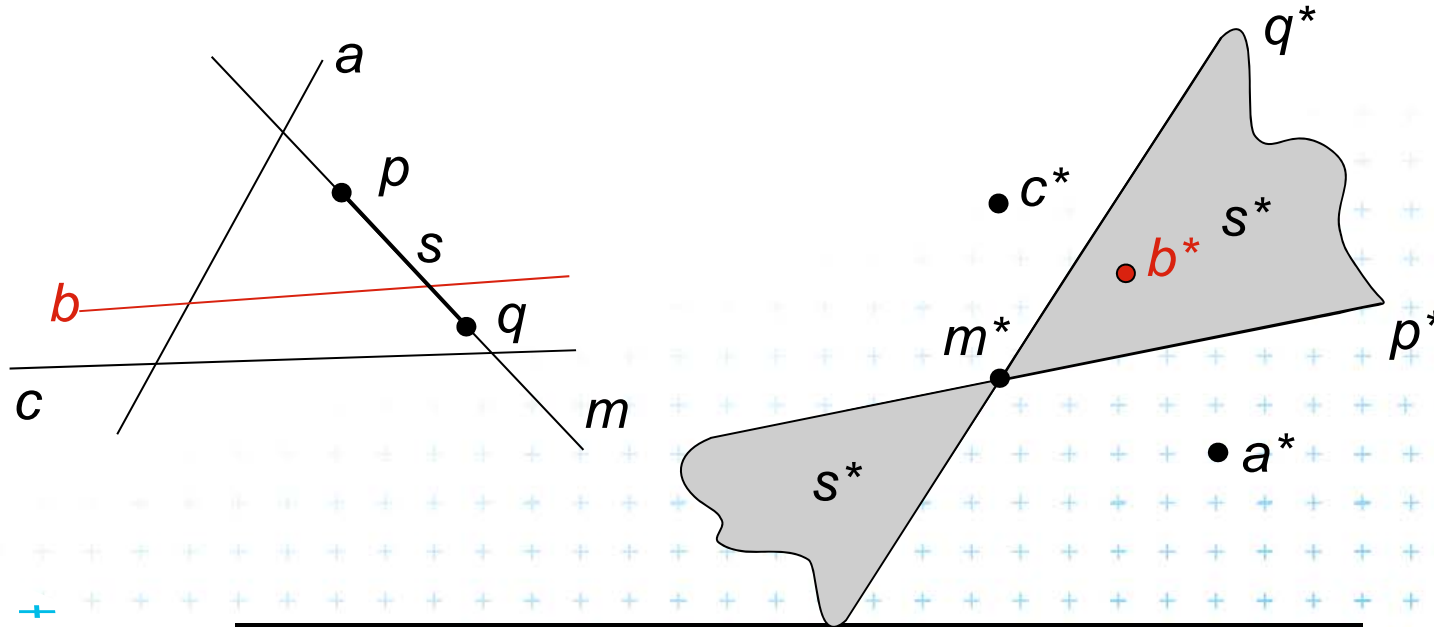
- Line segment  $s$

- = set of collinear points  $\xrightarrow{\text{dual}}$  set of lines passing one point
- union of these lines is a (left-right) **double wedge**  $s^*$



# Intersection of line and line segment

- Line  $b$  intersects line segment  $s$ 
  - if point  $b^*$  lays in the double wedge  $s^*$ ,  
i.e., between the duals  $p^*, q^*$  of segment endpoints  $p, q$
  - point  $p$  lies above line  $b$  and  $q$  lies below line  $b$
  - point  $b^*$  lies above line  $p^*$  and  $b^*$  lies below line  $q^*$





### 3. Polar duality (Polarity)

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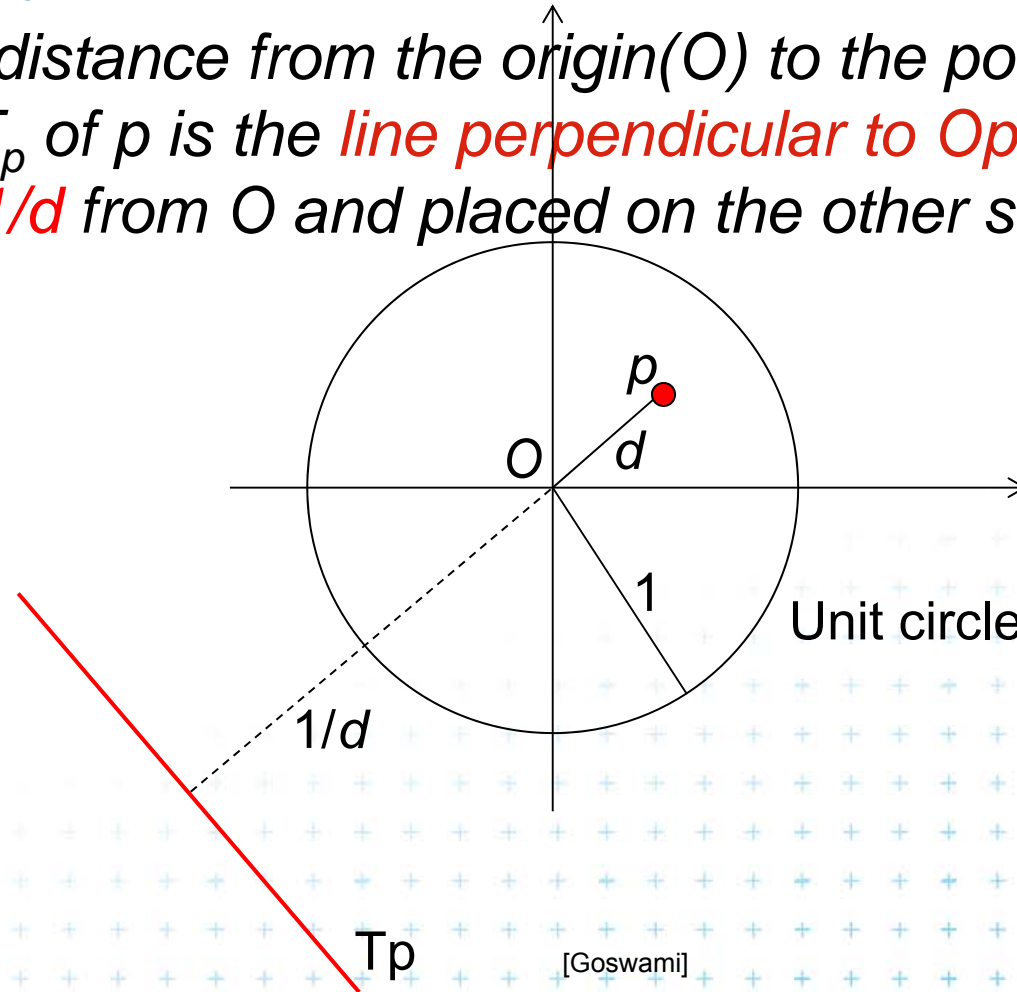
- Another example of **point-line duality**
- In 2D: Point  $p = (p_x, p_y)$  in the primal plane corresponds to a line  $T_p$  with equation  $ax + by = 1$  in the dual plane and vice versa  $p_x a + p_y b = 1$
- In dD: Point  $p$  is taken as a radius-vector (starts in origin  $O$ ). The **dot product**  $(p \cdot \mathbf{x}) = 1$  defines a **polar hyperplane**  $p^* = T_p = \{ \mathbf{x} \in R^d : (p \cdot \mathbf{x}) = 1 \}$
- Used in theory of polytopes



# Polar duality (Polarity)

- Geometrically in 2D, this means that

- if  $d$  is the distance from the origin( $O$ ) to the point  $p$ , the dual  $T_p$  of  $p$  is the **line perpendicular to  $Op$**  at distance  $1/d$  from  $O$  and placed on the other side of  $O$ .



# 4. Convex hull using duality – definitions

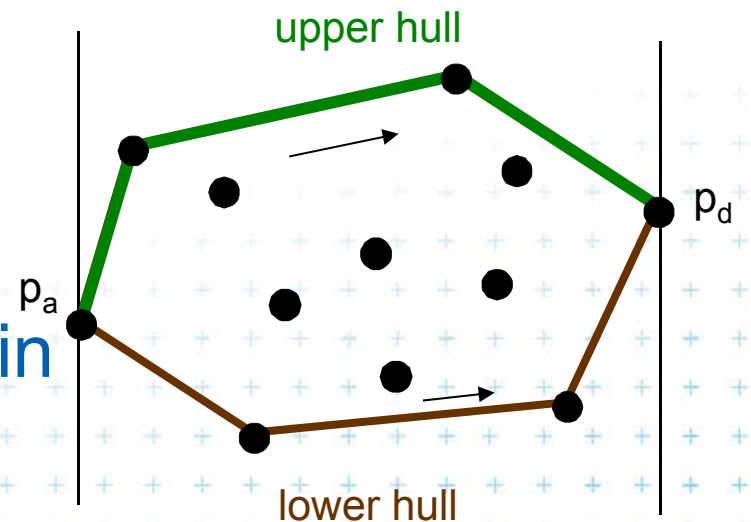
- An optimal algorithm
  - Let  $P$  be the given set of  $n$  points in the plane.
  - Let  $p_a \in P$  be the point with smallest x-coordinate
  - Let  $p_d \in P$  be the point with largest x-coordinate
- Both  $p_a$  and  $p_d \in CH(P)$

Upper hull = CW polygonal chain

$p_a, \dots, p_d$  along the hull

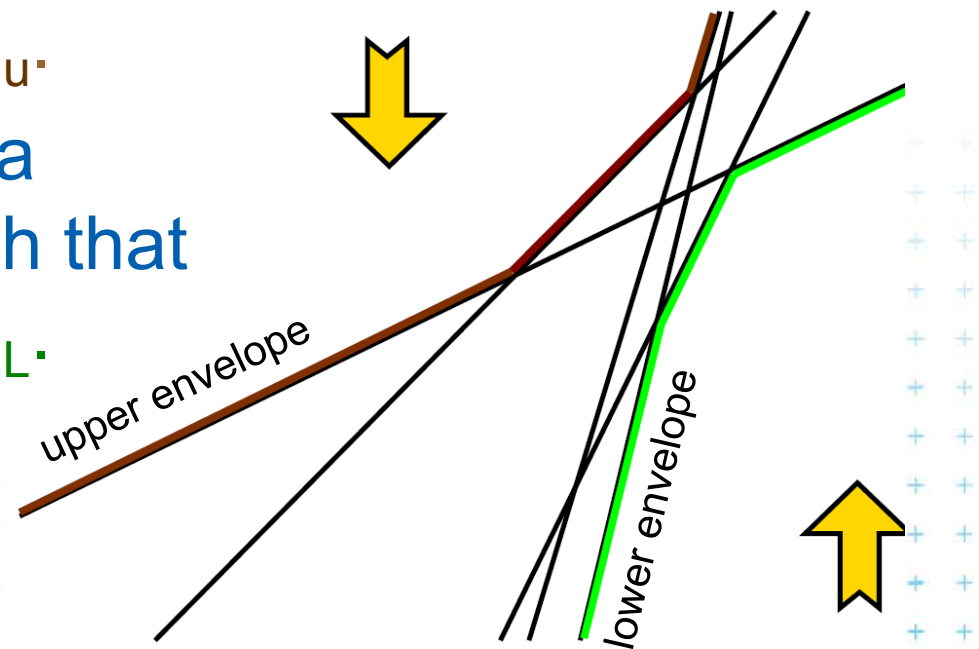
Lower hull = CCW polygonal chain

$p_a, \dots, p_d$  along the hull

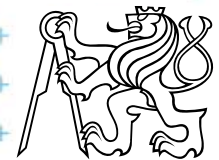


# Definitions

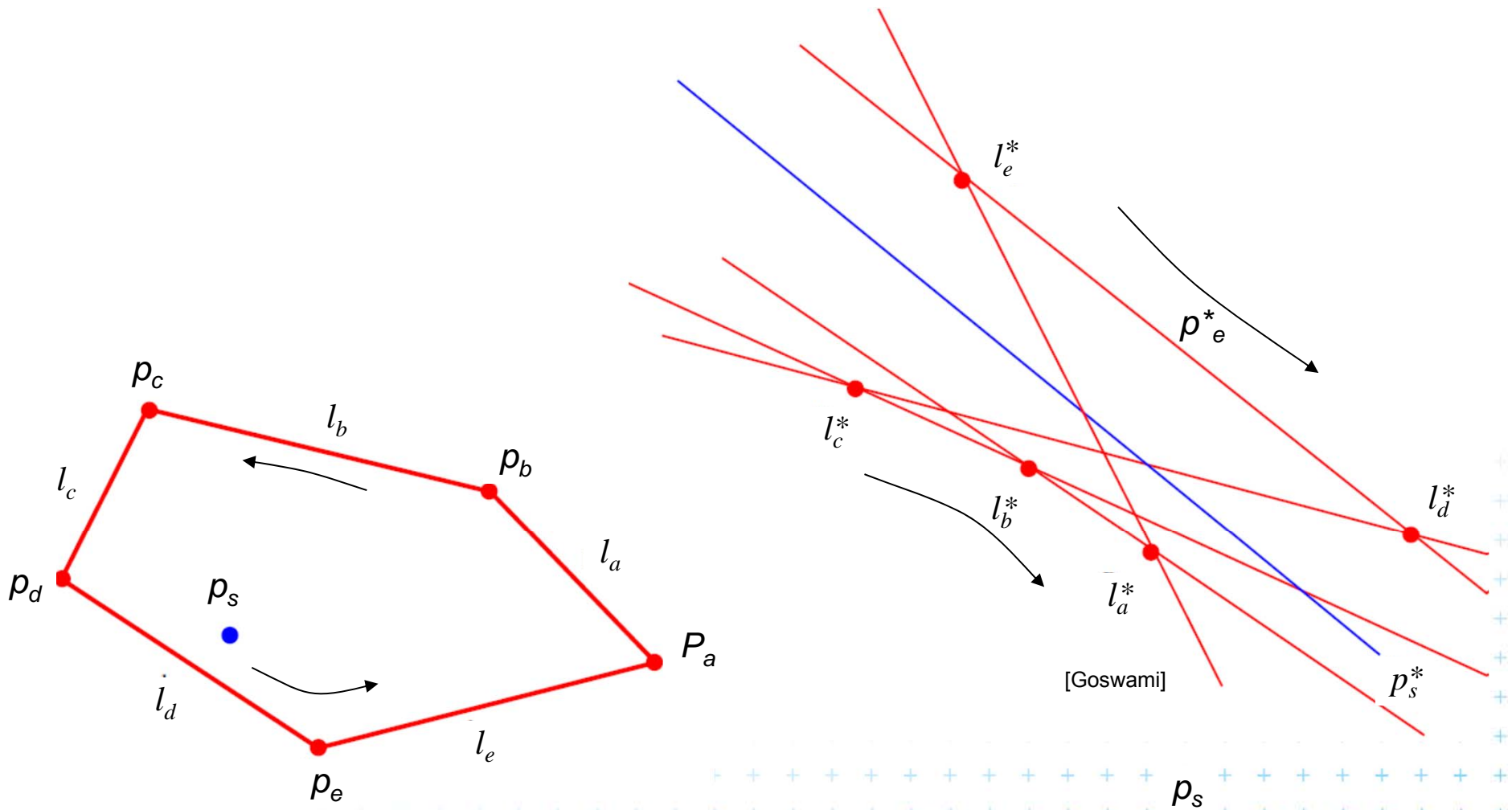
- Let  $L$  be a set of lines in the plane
- The upper envelope is a polygonal chain  $E_u$  such that no line  $l \in L$  is above  $E_u$ .
- The lower envelope is a polygonal chain  $E_L$  such that no line  $l \in L$  is below  $E_L$ .



[Goswami]

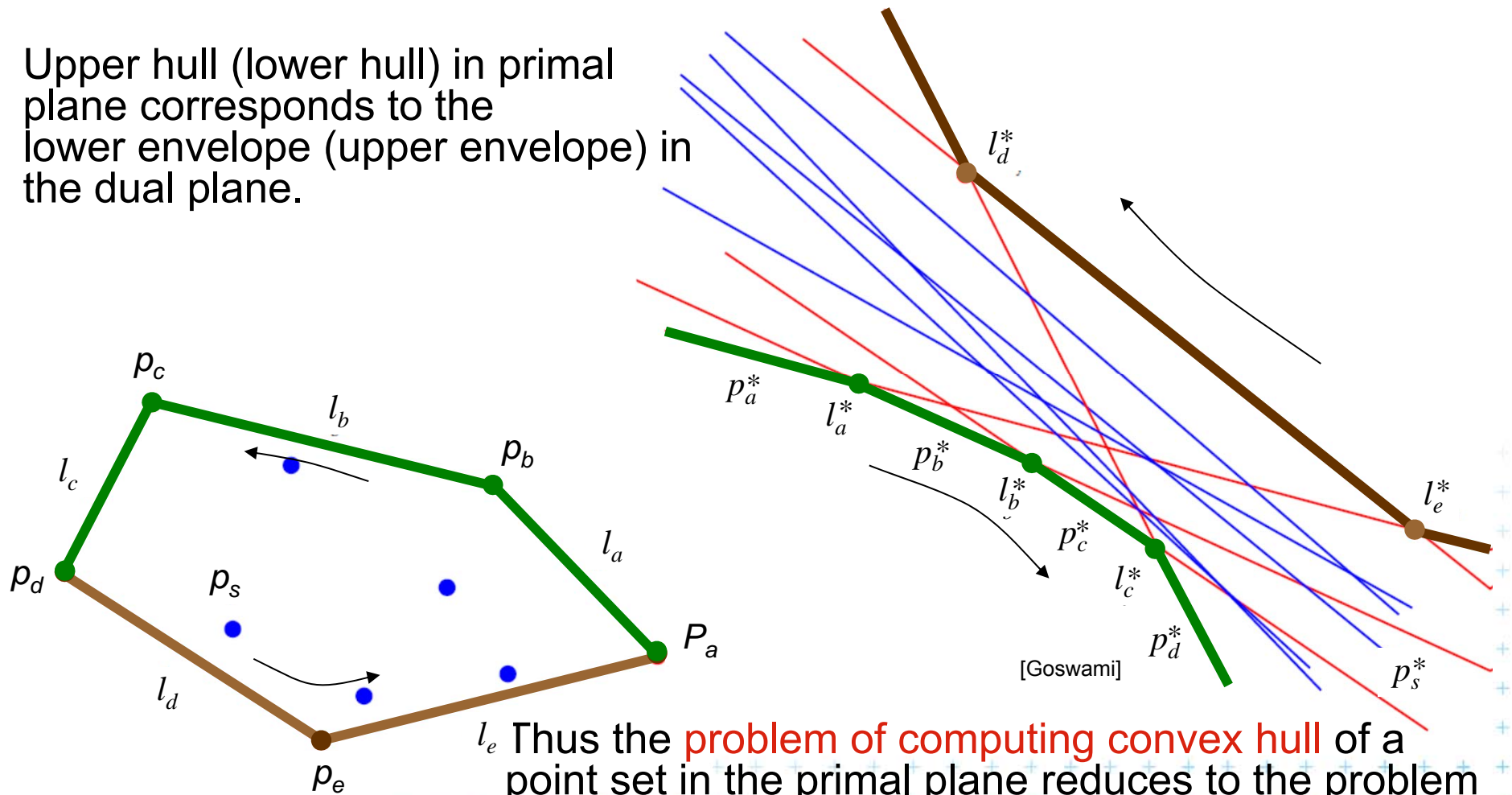


# Connection between Hull and Envelope



# Connection between Hull and Envelope

Upper hull (lower hull) in primal plane corresponds to the lower envelope (upper envelope) in the dual plane.



Thus the **problem of computing convex hull** of a point set in the primal plane reduces to the problem of computing **upper and lower envelopes** of the line set in the dual plane.



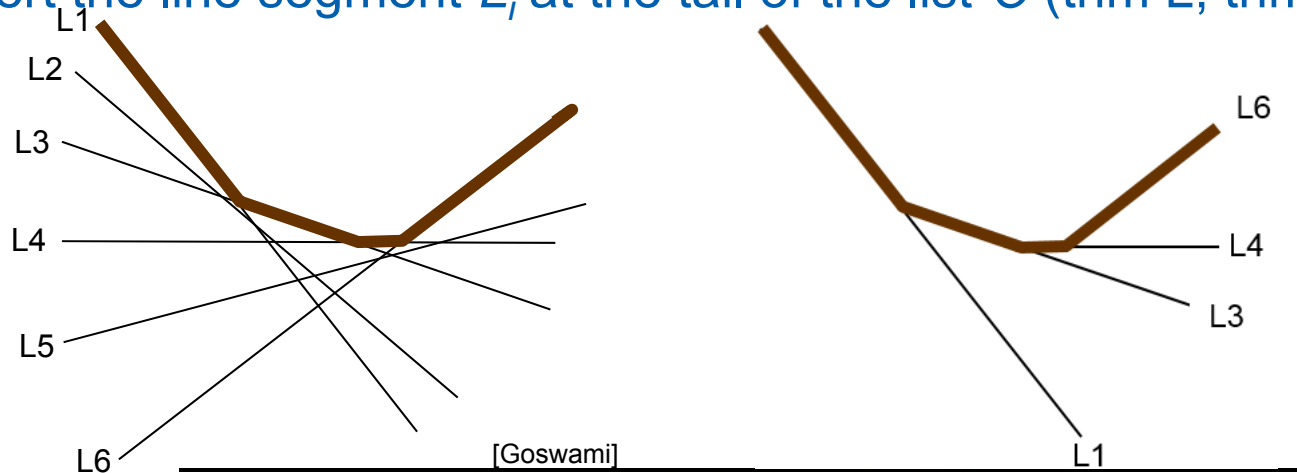
# Upper envelope algorithm

## UpperEnvelope( $L$ )

Set of lines  $L$  sorted by **increasing** order of slopes ( $-90^\circ$  to  $90^\circ$ )

Polygonal chain  $O$  representing the upper hull

1.  $O = L_1$  // the only complete line in  $O$
2. for  $i = 2$  to  $n$
3.  $L =$  last entry in  $O$  //  $O$  contains half-lines, or line segments, // except of complete line  $L_1$  at the beginning
4. while( the line segment  $L$  does not intersect the new line  $L_i$ )
5. remove  $L$  from  $O$  and replace  $L$  with its predecessor //  $L_2, L_5$
6. insert the line segment  $L_i$  at the tail of the list  $O$  (trim  $L$ , trim  $L_i$ )



# Convex hull via upper and lower envelope

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## ■ Upper envelope complexity

- After sorting  $n$  lines by their slopes in  $O(n \log n)$  time, the upper envelope can be obtained in  $O(n)$  time
- Proof: It may check more than one line segment when inserting a new line, but those ones **checked are all removed except the last one**.  
( $O(n)$  insertions, max  $O(n)$  removals  
=>  $O(n)$  all steps. Average step  $O(1)$  amortized time)

## ■ Convex hull complexity

- Given a set  $P$  of  $n$  points in the plane,  $\text{CH}(P)$  can be computed in  **$O(n \log n)$  time using  $O(n)$  space**.





# Applications of line arrangement

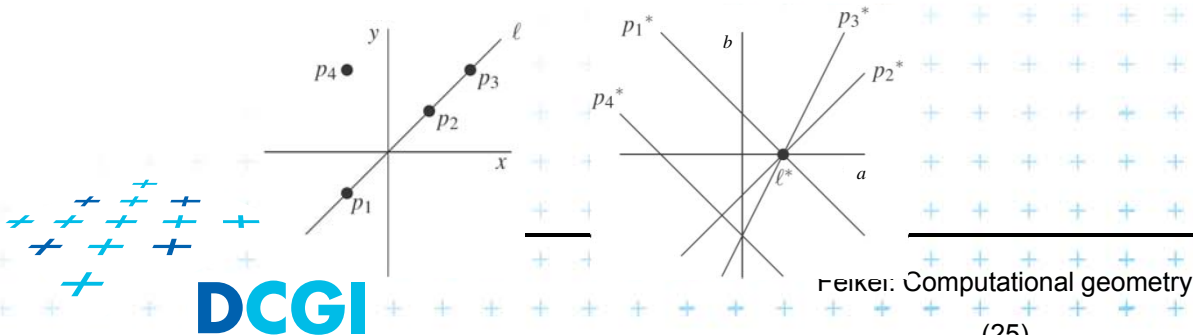
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Examples of applications – solved in  $O(n^2)$  and  
↙  $O(n^2)$  space by constructing a line arrangement or  
 $O(n)$  space through topological plain sweep.

## a) General position test:

Given a set of  $n$  points in the plane, determine whether any three are **collinear**.

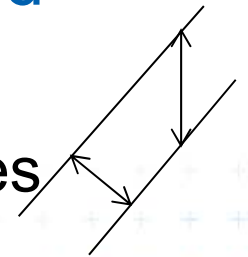
- Construct an arrangement in dual plane
- Report intersections of more than 2 lines



## b) Minimum k-corridor

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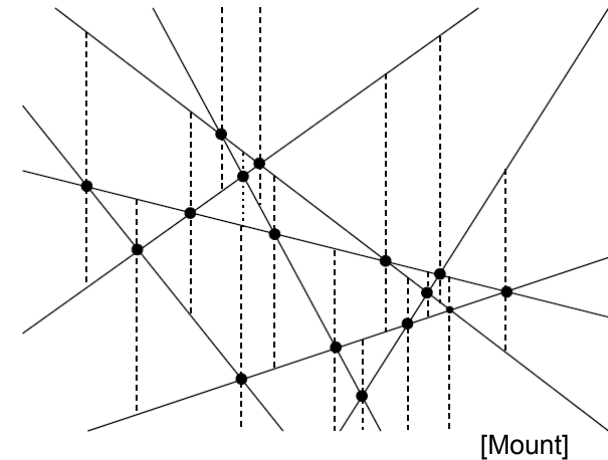
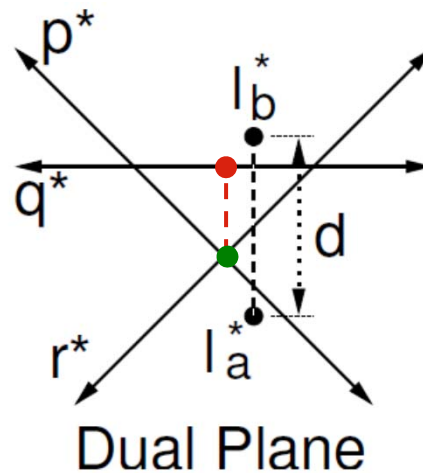
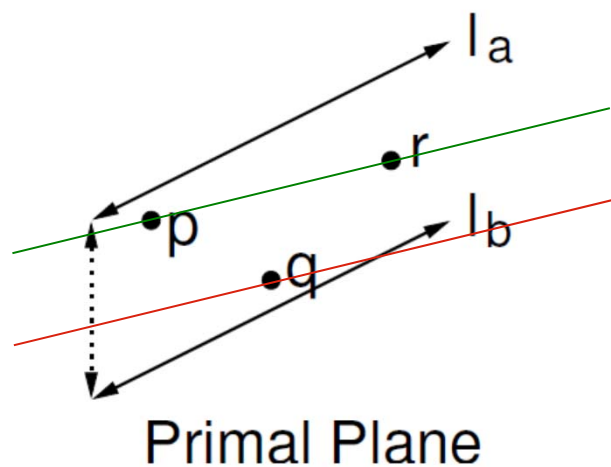
- Given a set of  $n$  points, and an integer  $k \in [1 : n]$ , determine the **narrowest pair of parallel lines** that **enclose at least  $k$  points** of the set.
- The distance between the lines can be defined
  - either as the **vertical distance** between the lines
  - or as the **perpendicular distance** between the lines
- Simplifications
  - Assume  $k = 3$  and **no 3 points are collinear**  
=> narrowest corridor - contains exactly 3 points  
- has width  $> 0$
  - No 2 points have the same x coordinate (avoid I duals)



vertical



## b) Minimum k-corridor



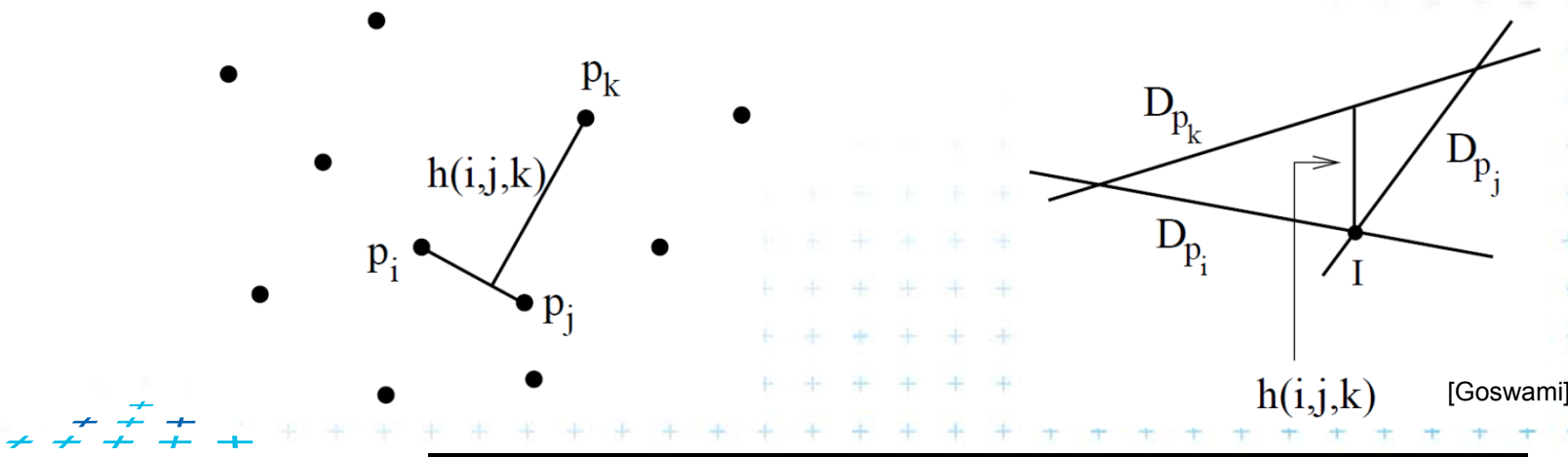
- **Vertical distance** of  $l_a, l_b = (-)$  distance of  $l_a^*, l_b^*$
- Nearest lines – one passes 2 vertices, e.g.,  $p$  &  $r$
- In dual plane are represented as intersection  $p^* \times r^*$
- Find nearest 3-stabber similarly as trapezoidal map
- $O(n^2)$  time and  $O(n)$  space – topological line sweep



## c) Minimum area triangle

[Goswami]

- Given a set of  $n$  points in the plane, determine the minimum area triangle whose vertices are selected from these points
- Construct “trapezoids” as in the nearest corridor
- Minimize **perpendicular distances** (converted from vertical) **multiplied** by the distance from  $p_i$  to  $p_j$



## d) Sorting all angular sequences – naïve

- Natural application of duality and arrangements
- Important for **visibility graph** computation
- Set of  $n$  points in the plane
- For **each point** perform an **CCW angular sweep**
- Naïve: for each point compute angles to remaining  $n - 1$  points and sort them
- $\Rightarrow O(n \log n)$  time per point
- $O(n^2 \log n)$  time overall
- Arrangements can get rid of  $O(\log n)$  factor



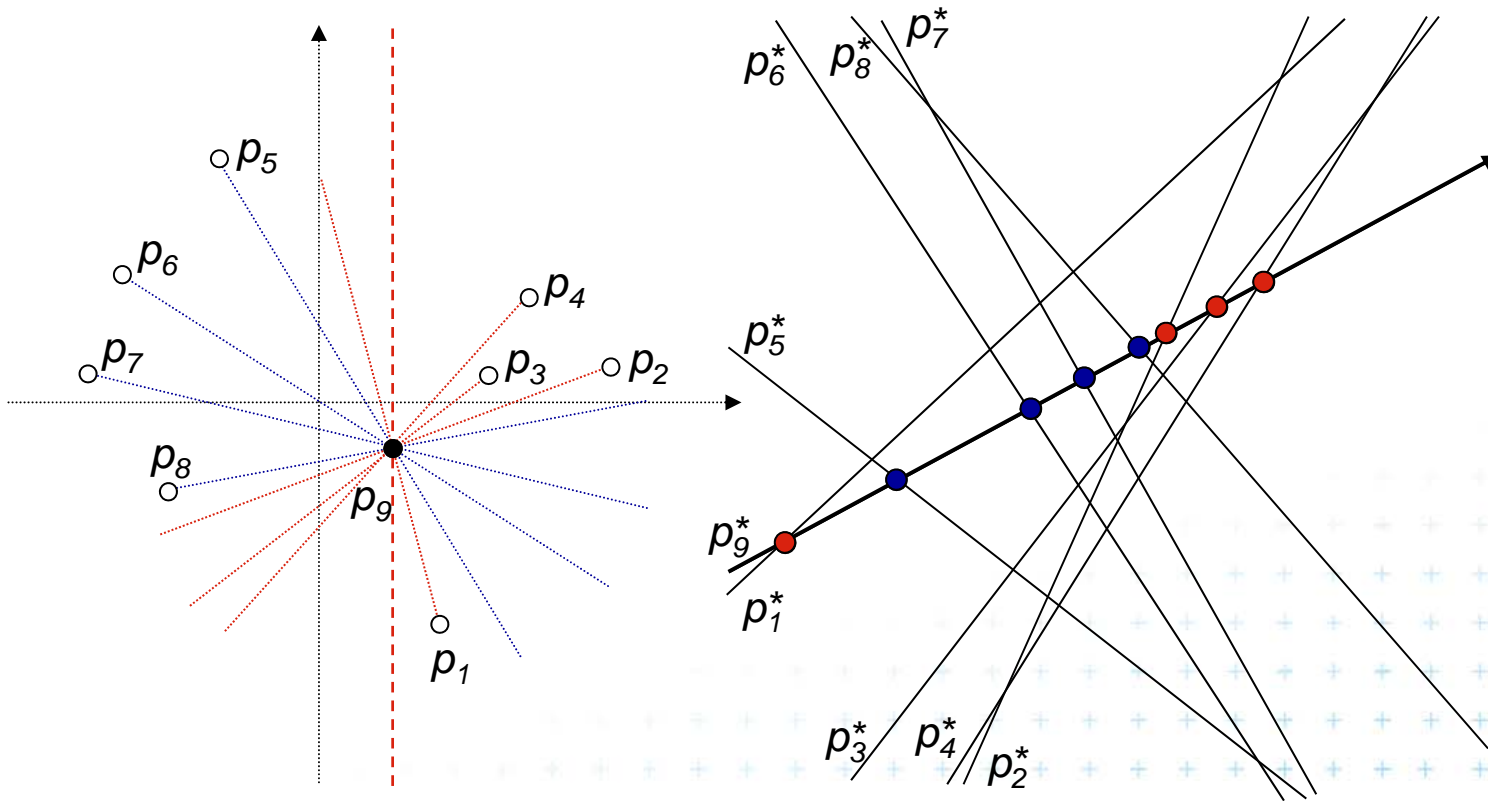
## d) Sorting all angular sequences – optimal

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- For point  $p_i$ 
  - Dual of point  $p_i$  is line  $p_i^*$
  - Line  $p_i^*$  intersects other dual lines in **order of slope**  
(angles from  $-90^\circ$  to  $90^\circ$ ) (180°)
  - We need **order of angles around  $p_i$**   
(angles from  $-90^\circ$  to  $270^\circ$ ) (360°)
  - Split points in primal plane by vertical line through  $p_i$
  - First, report intersections of points **right of  $p_i$**
  - Second, report the intersections of points **left of  $p_i$**
  - Once the arrangement is constructed:  
 $O(n)$  time for point,  **$O(n^2)$  time for all  $n$  points**



# d) Angular sequence around $p_9$



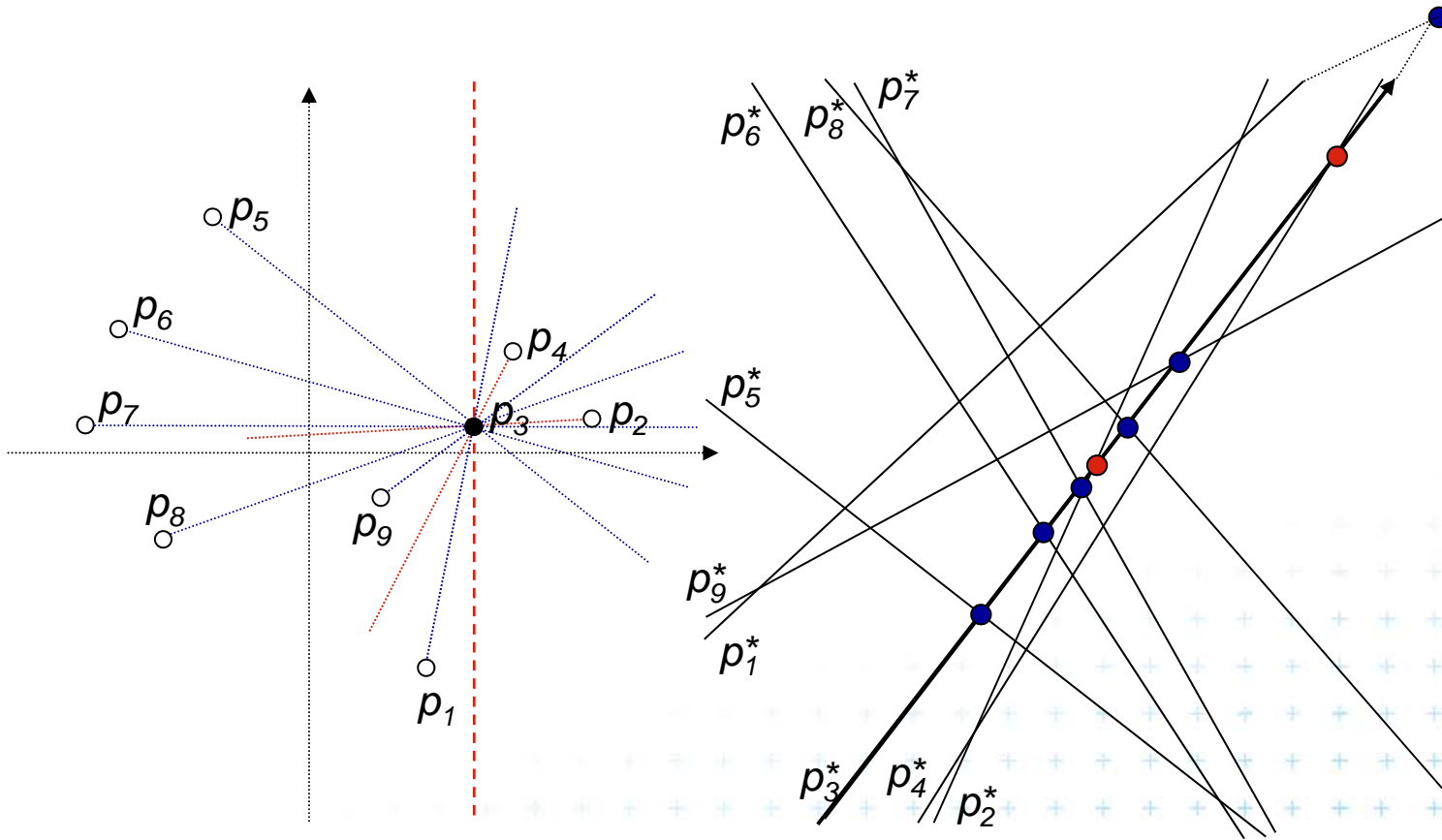
In primal plane

In dual plane

Point order around  $p_9$ :  $p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8$



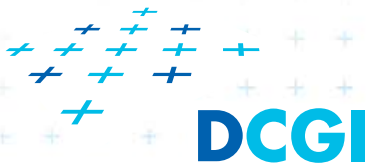
# d) Angular sequences around $p_3$



In primal plane

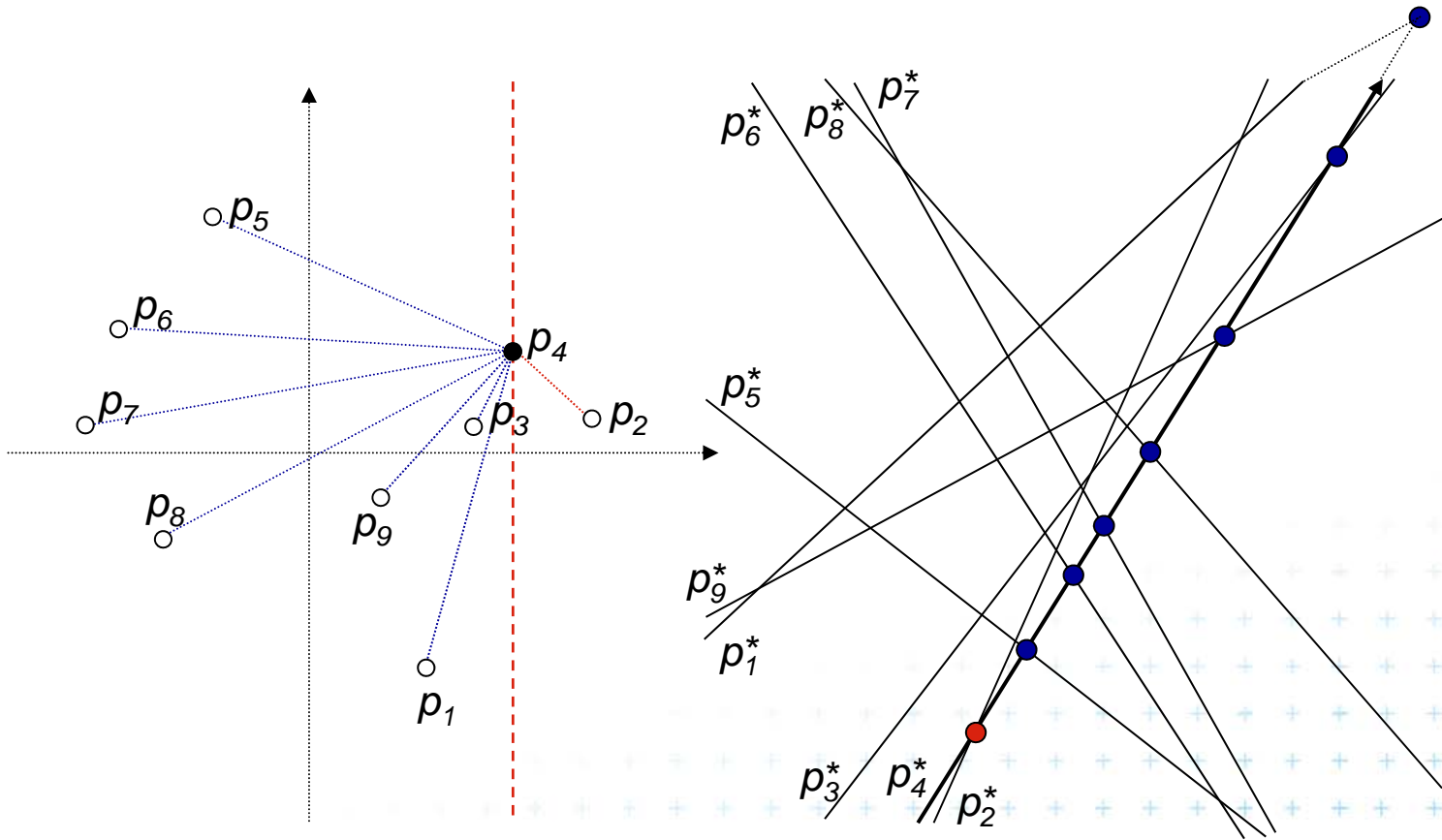
In dual plane

Point order around  $p_3$  :  $p_2, p_4, p_5, p_6, p_7, p_8, p_3, p_1$





# d) Angular sequences around $p_4$



In primal plane

In dual plane

Point order around  $p_4$  :  $p_2, p_5, p_6, p_7, p_8, p_9, p_3, p_1$



## e) More applications of line arrangement

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### Visibility graph

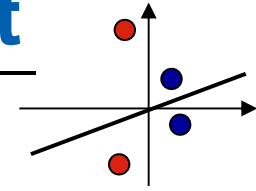
Given a set of  $n$  non-intersecting **line segments**, compute the *visibility graph*, whose vertices are the endpoints of the segments, and whose edges are pairs of visible endpoints (use angular sequences).

### Maximum stabbing line

Given a set of  $n$  line segments in the plane, compute the line that stabs (intersects) the maximum number of these line segments.



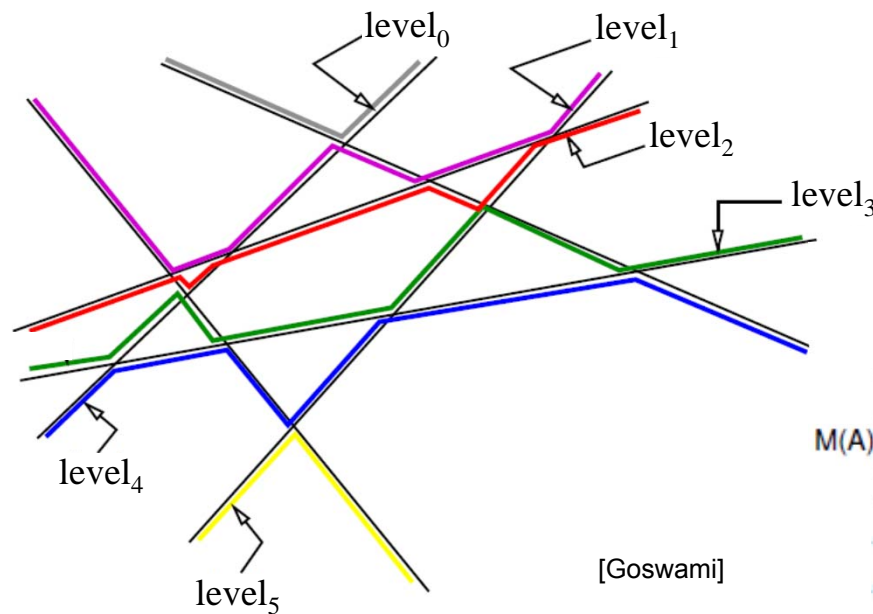
# More applications of line arrangement



## Ham-Sandwich cut

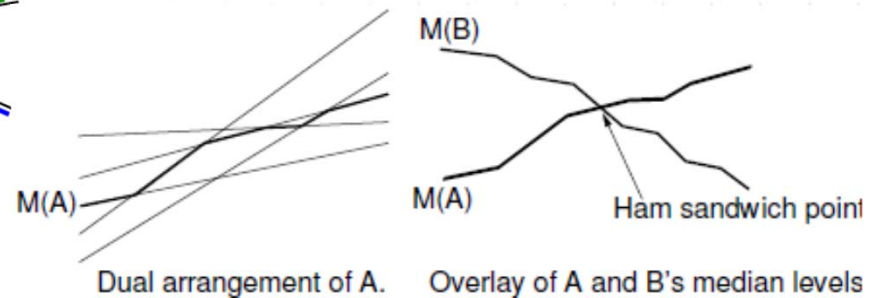
Given two sets of points,  $n$  red and  $m$  blue points compute a **single line that simultaneously bisects both sets**

Principle – intersect middle levels of arrangements



[Goswami]

Point at  $k$ -th level  $L_k$  has at most  $k$  lines above and at most  $n - k - 1$  lines below



[Mount]



# References

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- [Berg] Mark de Berg, Otfried Cheong, Marc van Kreveld, Mark Overmars: Computational Geometry: Algorithms and Applications, Springer-Verlag, 3rd rev. ed. 2008. 386 pages, 370 fig. ISBN: 978-3-540-77973-5, Chapters 8., <http://www.cs.uu.nl/geobook/>
- [Mount] Mount, D.: *Computational Geometry Lecture Notes for Fall 2016*, University of Maryland, Lectures 7, 14, 15, 27 and 28. <http://www.cs.umd.edu/class/fall2016/cmsc754/Lects/cmsc754-fall16-lects.pdf>
- [applet] Allen K. L. Miu: Duality Demo <http://nms.lcs.mit.edu/~aklmiu/6.838/dual/>
- [Goswami] Partha P. Goswami: Duality Transformation and its Application to Computational Geometry, University of Calcutta, India <http://www.tcs.tifr.res.in/~igga/lectureslides/partha-lec-iisc-jul09.pdf>

