Statistical Machine Learning (BE4M33SSU)
Lecture 11: Hidden Markov Models

Czech Technical University in Prague

- Hidden Markov Models
- Inference algorithms for HMMs
- Parameter learning for HMMs

1. Hidden Markov Models

- Let $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$ denote a sequence of hidden states from a finite set K.
- Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ denote a sequence of features from some feature space $\mathcal{X}$.

Definition 1. A joint p.d. on $\mathcal{X}^n \times K^n$ is a Hidden Markov model if

(a) the prior p.d. $p(\boldsymbol{s})$ for the sequences of hidden states is a Markov model, and

(b) the conditional distribution $p(\boldsymbol{x} \,|\, \boldsymbol{s})$ for the feature sequence is independent, i.e.

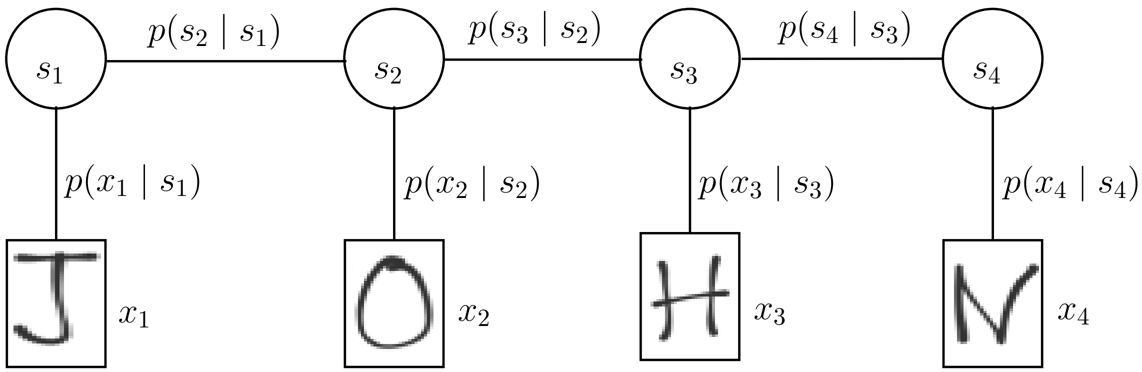
$$p(\boldsymbol{x} \,|\, \boldsymbol{s}) = \prod_{i=1}^{n} p(x_i \,|\, s_i).$$

The joint model is given by

$$p(\boldsymbol{x}, \boldsymbol{s}) = p(s_1) \prod_{i=2}^{n} p(s_i \,|\, s_{i-1}) \prod_{i=1}^{n} p(x_i \,|\, s_i).$$

1. Hidden Markov Models

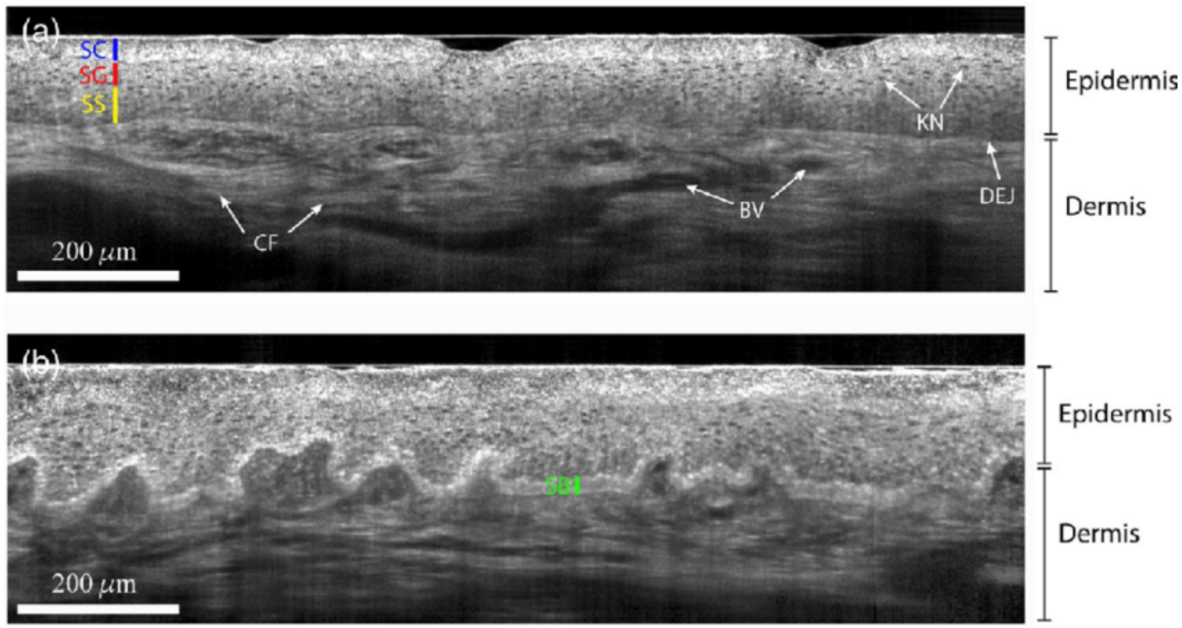
Example 1 (Text recognition, OCR).



- $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ – sequence of images with characters,
- $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$ – sequence of alphabetic characters,
- $p(s_i \,|\, s_{i-1})$ – language model,
- $p(x_i \,|\, s_i)$ – appearance model for characters.

2. Hidden Markov Models

Example 2 (Skin layer segmentation, OCT).



- $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ – sequence of image columns (features),
- $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$ – sequence of boundary height values,
- $p(s_i \,|\, s_{i-1})$ – boundary model,
- $p(x_i \,|\, s_i)$ – appearance model for image columns.

3. Inference algorithms for HMMs

How to find the most probable sequence of hidden states given the sequence of features $\boldsymbol{x}$

$$\boldsymbol{s}^* \in \arg\max_{\boldsymbol{s}\in K^n} \; p(s_1) \prod_{i=2}^{n} p(s_i \,|\, s_{i-1}) \prod_{i=1}^{n} p(x_i \,|\, s_i)$$

Take the logarithm, with

$$g_1(s_1, x_1) = \log[p(s_1) p(x_1 \,|\, s_1)]$$
$$g_i(s_{i-1}, s_i, x_i) = \log[p(s_i \,|\, s_{i-1}) p(x_i \,|\, s_i)],\; i = 2, \ldots, n$$

Solve the task

$$\boldsymbol{s}^* \in \arg\max_{\boldsymbol{s}\in K^n} \Big[g_1(s_1, x_1) + \sum_{i=2}^{n} g_i(s_{i-1}, s_i, x_i)\Big]$$

by dynamic programming as before for Markov models.

3. Inference algorithms for HMMs

How to compute marginal probabilities for hidden states given the sequence of features?

We want to compute $p(s_i = k, \boldsymbol{x})$, $\forall k \in K$ in some sequence position j.

$$p(s_j, \boldsymbol{x}) = \sum_{s_1 \in K} \cdots \cancel{\sum_{s_j \in K}} \cdots \sum_{s_n \in K} p(s_1) p(x_1 \,|\, s_1) \prod_{i=2}^{n} \big[p(s_i \,|\, s_{i-1}) p(x_i \,|\, s_i) \big]$$

This is now more complicated, because we need to sum over the leading and trailing hidden state variables. Do this by dynamic matrix-vector multiplication from the left and from the right.

Initialise $\phi_1(s_1) = p(s_1) p(x_1 \,|\, s_1)$ and $\psi_n(s_n) \equiv 1$ and recursively compute

$$\phi_i(s_i) = \sum_{s_{i-1} \in K} p(x_i \,|\, s_i)\, p(s_i \,|\, s_{i-1})\, \phi_{i-1}(s_{i-1}) \quad \forall s_i \in K$$

$$\psi_i(s_i) = \sum_{s_{i+1} \in K} p(x_{i+1} \,|\, s_{i+1})\, p(s_{i+1} \,|\, s_i)\, \psi_{i+1}(s_{i+1}) \quad \forall s_i \in K$$

Denoting the transition probability matrices by $P(i)$, we can write this equivalently as matrix-vector multiplications: $\boldsymbol{\phi}_i = P(i) \boldsymbol{\phi}_{i-1}$ and $\boldsymbol{\psi}_i = P^T(i+1) \boldsymbol{\psi}_{i+1}$.

3. Inference algorithms for HMMs

The marginal probabilities are then obtained from

$$p(s_i = k, \boldsymbol{x}) = \phi_i(s_i = k)\psi_i(s_i = k)$$

The computational complexity for computing *all* marginal probabilities in *all* positions $i = 1, \dots, n$ is thus $\mathcal{O}(nK^2)$.

Remark 1.

- We can also compute *pairwise* marginal probabilities from the ϕ-s and ψ-s

$$p(s_{i-1}, s_i, \boldsymbol{x}) = \phi_{i-1}(s_{i-1})\big[p(s_i \,|\, s_{i-1})p(x_i \,|\, s_i)\big]\psi_i(s_i)$$

- Computing conditional marginal probabilities is then easy:

$$p(s_i = k \,|\, \boldsymbol{x}) = \frac{p(s_i = k, \boldsymbol{x})}{\sum_{k' \in K} p(s_i = k', \boldsymbol{x})}$$

- The same holds for computing the probability that the model will generate the sequence of features $\boldsymbol{x}$: $p(\boldsymbol{x}) = \sum_{k \in K} p(s_i = k, \boldsymbol{x})$.

4. Learning algorithms for HMMs

Supervised learning:

Given i.i.d. training data $\mathcal{T}^m = \{(\boldsymbol{x}^j, \boldsymbol{s}^j) \in \mathcal{X}^n \times K^n \,|\, j = 1, \ldots, m\}$, estimate the parameters of the HMM by the maximum likelihood estimator.

This is done by simple "counting" as before for Markov models.

- Denote by $a_i(s_{i-1} = \ell, s_i = k)$ the number of examples in $\mathcal{T}^m$ for which $s_{i-1} = \ell$ and $s_i = k$.
- Denote by $b_i(s_i = k, x_i = x)$ the number of examples in $\mathcal{T}^m$ for which $s_i = k$ and $x_i = x$.

The estimates for the model parameters are then given by

$$p(s_i = k \,|\, s_{i-1} = \ell) = \frac{a_i(s_{i-1} = \ell, s_i = k)}{\sum_{k'} a_i(s_{i-1} = \ell, s_i = k')}, \; p(x_i = x \,|\, s_i = k) = \frac{b_i(s_i = k, x_i = x)}{\sum_{x'} b_i(s_i = k, x_i = x')}$$

Remark 2. This is easy to generalise for the case that $\mathcal{X} = \mathbb{R}^m$ and $p(x_i \,|\, s_i)$ is from some parametric distribution family.

Remark 3. Learning HMMs by empirical risk minimisation has been discussed in Lecture 5. (Structured output SVMs).

4. Learning algorithms for HMMs

Unsupervised learning:

Given i.i.d. training data $\mathcal{T}^m = \{\boldsymbol{x}^j \in \mathcal{X}^n \,|\, j = 1, \ldots, m\}$, estimate the parameters of the HMM by the maximum likelihood estimator.

We apply the EM-algorithm (aka Baum-Welch algorithm): Initialise the model, then iterate

E-step Use the current model estimate and compute the pairwise marginal probabilities

$$\alpha_{\boldsymbol{x}}(s_i, s_{i-1}) = p(s_i, s_{i-1} \,|\, \boldsymbol{x}) \quad s_{i-1}, s_i \in K$$

for each training example $\boldsymbol{x} \in \mathcal{T}^m$ and all positions $i = 2, \ldots, n$. See Remark 1.

M-step Use the α-s as soft labels for computing the counts as in supervised learning

$$a_i(s_{i-1}, s_i) = \sum_{\boldsymbol{x} \in \mathcal{T}^m} \alpha_{\boldsymbol{x}}(s_i, s_{i-1}), \qquad b_i(s_i, x) = \sum_{\boldsymbol{x} \in \mathcal{T}(x_i = x)} \alpha_{\boldsymbol{x}}(s_i)$$

where $\mathcal{T}(x_i = x)$ is the set of training examples with $x_i = x$ and $\alpha_{\boldsymbol{x}}(s_i)$ is given by $\alpha_{\boldsymbol{x}}(s_i) = \sum_{s_{i-1}} \alpha_{\boldsymbol{x}}(s_i, s_{i-1})$.

Summary

- Hidden Markov models are statistical models for pairs of processes (sequences) $(\boldsymbol{s}, \boldsymbol{x})$, where $\boldsymbol{s}$ is a sequence of hidden states and not directly observable.
- Markov models and HMMs are exponential families.
- Their parameters can be estimated by supervised leaning using either Maximum likelihood estimates or Empirical risk minimisation.
- Their parameters can be estimated by unsupervised learning using the EM-algorithm.
- All important inference and learning algorithms for HMMs have time complexity linear in the length of the sequence and quadratic in the size of the hidden state space.
- What if the space of hidden states is very large, e.g. $s_i \in \mathbb{Z}^k$? We can use recurrent neural networks for modelling $p(s_i \,|\, s_{i-1})$, e.g. Gated recurrent Units (GRU) or Long short-term memory models (LSTM).