Statistical Machine Learning (BE4M33SSU)
Lecture 10: Markov Models

Czech Technical University in Prague

- Markov models on sequences
- Inference algorithms for Markov models
- Parameter learning for Markov models

1. Structured hidden states

Models discussed so far: mainly classifiers predicting a categorical (class) variable $y \in \mathcal{Y}$

Often in applications: the hidden state y is a structured variable.

Here: the hidden state y is given by a **sequence** of categorical variables.

Application examples:

- text recognition (printed, handwritten, “in the wild”),
- speech recognition (single word recognition, continuous speech recognition, translation),
- robot self localisation.

Markov Models and Hidden Markov Models on chains:
a class of generative probabilistic models for sequences of features and sequences of categorical variables.

2. Markov Models

Let $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$ denote a sequence of length n with elements from a finite set K.

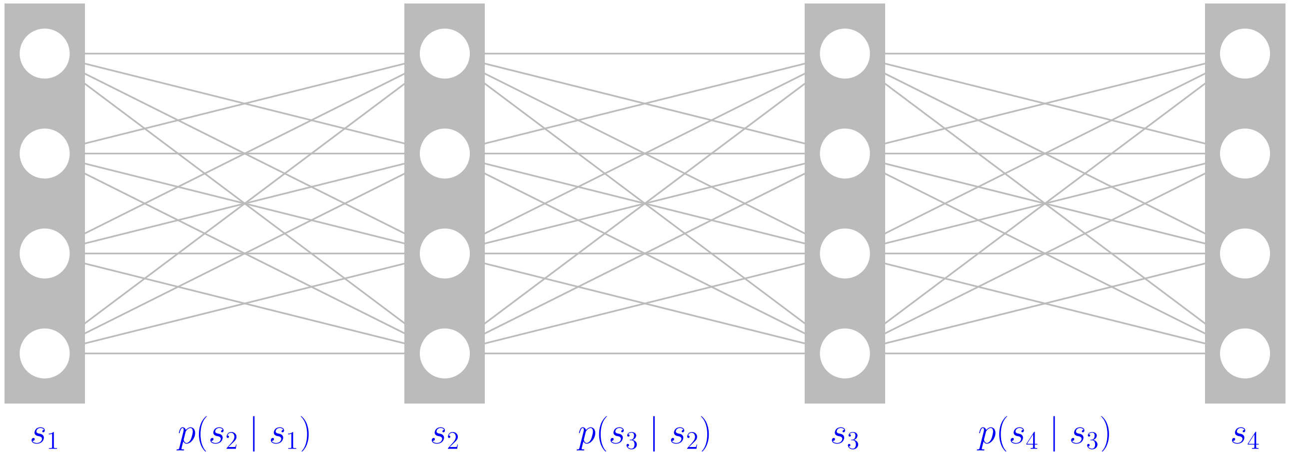
Any joint probability distribution on K^n can be written as

$$p(s_1, s_2, \ldots, s_n) = p(s_1)\, p(s_2 \,|\, s_1)\, p(s_3 \,|\, s_2, s_1) \cdot \ldots \cdot p(s_n \,|\, s_1, \ldots, s_{n-1})$$

Definition 1. A joint p.d. on K^n is a Markov model if

$$p(\boldsymbol{s}) = p(s_1)\, p(s_2 \,|\, s_1)\, p(s_3 \,|\, s_2) \cdot \ldots \cdot p(s_n \,|\, s_{n-1}) = p(s_1) \prod_{i=2}^{n} p(s_i \,|\, s_{i-1})$$

holds for any $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$.



2. Markov Models

Example 1 (Random walk on a graph)**.**

- Let (V, E) be a directed graph. A random walk in (V, E) is described by a sequence $\boldsymbol{s} = (s_1, \ldots, s_t, \ldots)$ of visited nodes, i.e. $s_t \in V$.
- The walker starts in node $i \in V$ with probability $p(s_1 = i)$.
- The edges of the graph are weighted by $w \colon E \to \mathbb{R}_+$, s.t.

$$\sum_{j \colon (i,j) \in E} w_{ij} = 1 \quad \forall i \in V$$

- In the current position $s_t = i$, the walker randomly chooses an outgoing edge with probability given by the weights and moves along this edge, i.e.

$$p(s_{t+1} = j \,|\, s_t = i) = \begin{cases} w_{ij} & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

3. Algorithms: Computing the most probable sequence

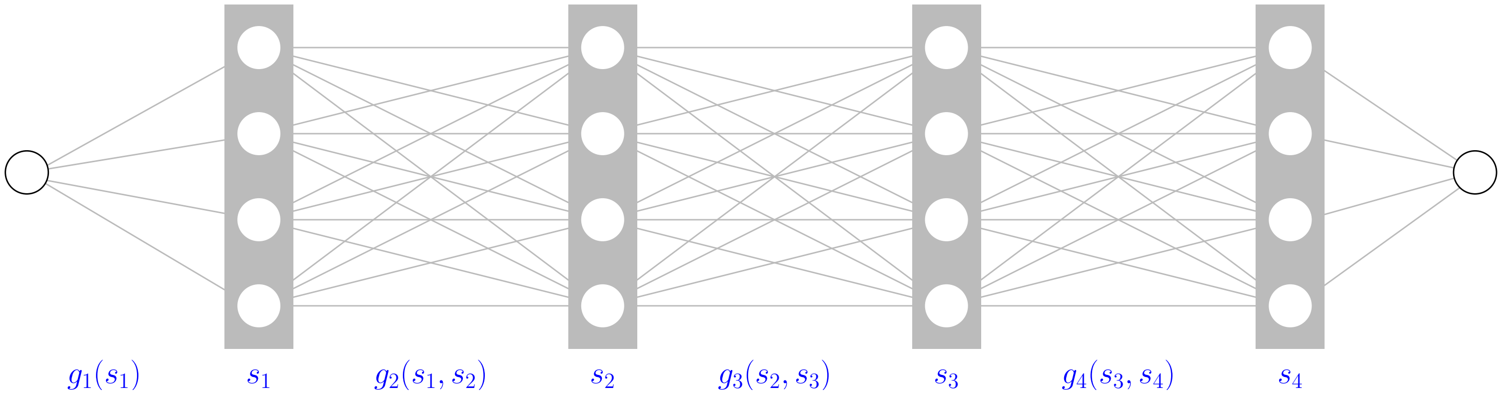
How to compute the most probable sequence $\boldsymbol{s}^* \in \arg\max_{\boldsymbol{s}\in K^n} \Big[p(s_1)\prod_{i=2}^{n} p(s_i \,|\, s_{i-1})\Big]$?

Take the logarithm of $p(\boldsymbol{s})$: $\boldsymbol{s}^* \in \arg\max_{\boldsymbol{s}\in K^n} \big[g_1(s_1) + \sum_{i=2}^{n} g_i(s_{i-1}, s_i)\big]$

and apply dynamic programming: Set $\phi_1(s_1) \equiv g_1(s_1)$ and compute

$$\phi_i(s_i) = \max_{s_{i-1}\in K} \big[\phi_{i-1}(s_{i-1}) + g_i(s_{i-1}, s_i)\big] \quad \forall s_i \in K.$$

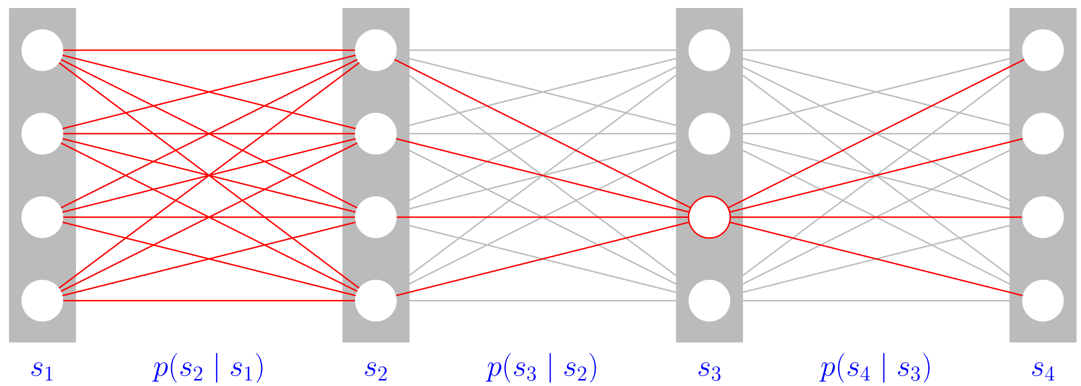
Finally, find $s_n^* \in \arg\max_{s_n\in K} \phi_n(s_n)$ and back-track the solution. This corresponds to searching the best path in the graph



3. Algorithms: Computing marginal probabilities

How to compute marginal probabilities for the sequence element s_j in position j

$$p(s_j) = \sum_{s_1 \in K} \cdots \cancel{\sum_{s_j \in K}} \cdots \sum_{s_n \in K} p(s_1) \prod_{i=2}^{n} p(s_i \,|\, s_{i-1})$$



Summation over the trailing variables is easily done because:

$$\sum_{s_n \in K} p(s_1) \cdots p(s_{n-1} \,|\, s_{n-2})\, p(s_n \,|\, s_{n-1}) = p(s_1) \cdots p(s_{n-1} \,|\, s_{n-2})$$

The summation over the leading variables is done dynamically: Begin with $p(s_1)$ and compute

$$p(s_i) = \sum_{s_{i-1} \in K} p(s_i \,|\, s_{i-1})\, p(s_{i-1}) \quad \forall s_i \in K$$

3. Algorithms: Computing marginal probabilities

This computation is equivalent to a matrix vector multiplication: Consider the values $p(s_i = k \,|\, s_{i-1} = k')$ as elements of a matrix $P_{kk'}(i)$ and the values of $p(s_i = k')$ as elements of a vector $\boldsymbol{\pi}_i$. Then the computation above reads as $\boldsymbol{\pi}_i = P(i)\boldsymbol{\pi}_{i-1}$.

Remark 1.

- A Markov model is called *homogeneous* if the transition probabilities $p(s_i = k \,|\, s_{i-1} = k')$ do not depend on the position i in the sequence. In this case the formula $\boldsymbol{\pi}_i = P^{i-1}\boldsymbol{\pi}_1$ holds for the computation of the marginal probabilities.

- Notice that the preferred direction (from first to last) in the Def. 1 of a Markov model is only apparent. By computing the marginal probabilities $p(s_i)$ and by using $p(s_i \,|\, s_{i-1})p(s_{i-1}) = p(s_{i-1}, s_i) = p(s_{i-1} \,|\, s_i)p(s_i)$, we can rewrite the model in reverse order.

3. Algorithms: Learning a Markov model

Suppose we are given i.i.d. training data $\mathcal{T}^m = \{\boldsymbol{s}^j \in K^n \,|\, j = 1, \ldots, m\}$ and want to estimate the parameters of the Markov model by the maximum likelihood estimate. This is very easy:

- Denote by $\alpha(s_{i-1} = \ell, s_i = k)$ the number of sequences in $\mathcal{T}^m$ for which $s_{i-1} = \ell$ and $s_i = k$.
- The estimates for the conditional probabilities are then given by

$$p(s_i = k \,|\, s_{i-1} = \ell) = \frac{\alpha(s_{i-1} = \ell, s_i = k)}{\sum_k \alpha(s_{i-1} = \ell, s_i = k)}.$$

Proof (idea):

Consider all terms in the log-likelihood that depend on the transition probability from $(i-1) \to i$ and rewrite them using transition counts $\alpha(s_{i-1} = \ell, s_i = k)$

$$\frac{1}{m} \sum_{\boldsymbol{s} \in \mathcal{T}^m} \log p(s_i \,|\, s_{i-1}) = \frac{1}{m} \sum_{k,\ell \in K} \alpha(s_{i-1} = \ell, s_i = k) \log p(s_i = k \,|\, s_{i-1} = \ell)$$

Maximise this w.r.t. $p(s_i \,|\, s_{i-1})$ under the constraint $\sum_{s_i \in K} p(s_i \,|\, s_{i-1}) = 1$.

3. Algorithms: Learning a Markov model

Markov models are **exponential families**. For simplicity we show this for the family of homogeneous Markov models on sequences $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$ of length n under the additional assumption that $p(s_1) = \frac{1}{K}$.

We have

$$p(\boldsymbol{s}) = \frac{1}{K} \prod_{i=2}^{n} p(s_i \,|\, s_{i-1})$$

- sufficient statistic: $\Phi(\boldsymbol{s})$ is a $K \times K$ matrix with entries $\Phi_{kl}(\boldsymbol{s})$ counting the number of transitions from state l to state k in the sequence $\boldsymbol{s}$.
- natural parameter: H is a $K \times K$ matrix with entries $H_{kl} = \log p(s_i = k \,|\, s_{i-1} = l)$

We can write the probability of sequences as

$$p(\boldsymbol{s}; H) = \exp\big[\langle \Phi(\boldsymbol{s}), H \rangle - \log(K)\big]$$

Remark 2. This can be generalised for models with non-uniform $p(s_1)$ and also for general (i.e. non-homogeneous) Markov models.

4. Return times and limiting distributions

- A homogeneous Markov model is *irreducible* if each state l can be reached starting from any state k with non-zero probability (after some number of transitions).
- A state k has *return time* τ if it can be reached with non-zero probability after τ transitions when starting from itself.
- A state $k \in K$ is *a-periodic* if the greatest common divisor of its return times is 1.

Theorem 1. *Let P be the transition probability matrix of an irreducible homogeneous Markov model with a-periodic states. Then there exists a unique marginal probability vector $\boldsymbol{\pi}^*$ s.t. $P\boldsymbol{\pi}^* = \boldsymbol{\pi}^*$. Moreover, it is a limiting distribution, i.e.*

$$\lim_{t\to\infty} P^t \boldsymbol{\pi} = \boldsymbol{\pi}^*$$

for arbitrary starting distributions $\boldsymbol{\pi}$.

Q: What conditions on the graph in Example 1 ensure that this theorem applies for the random walk considered there?