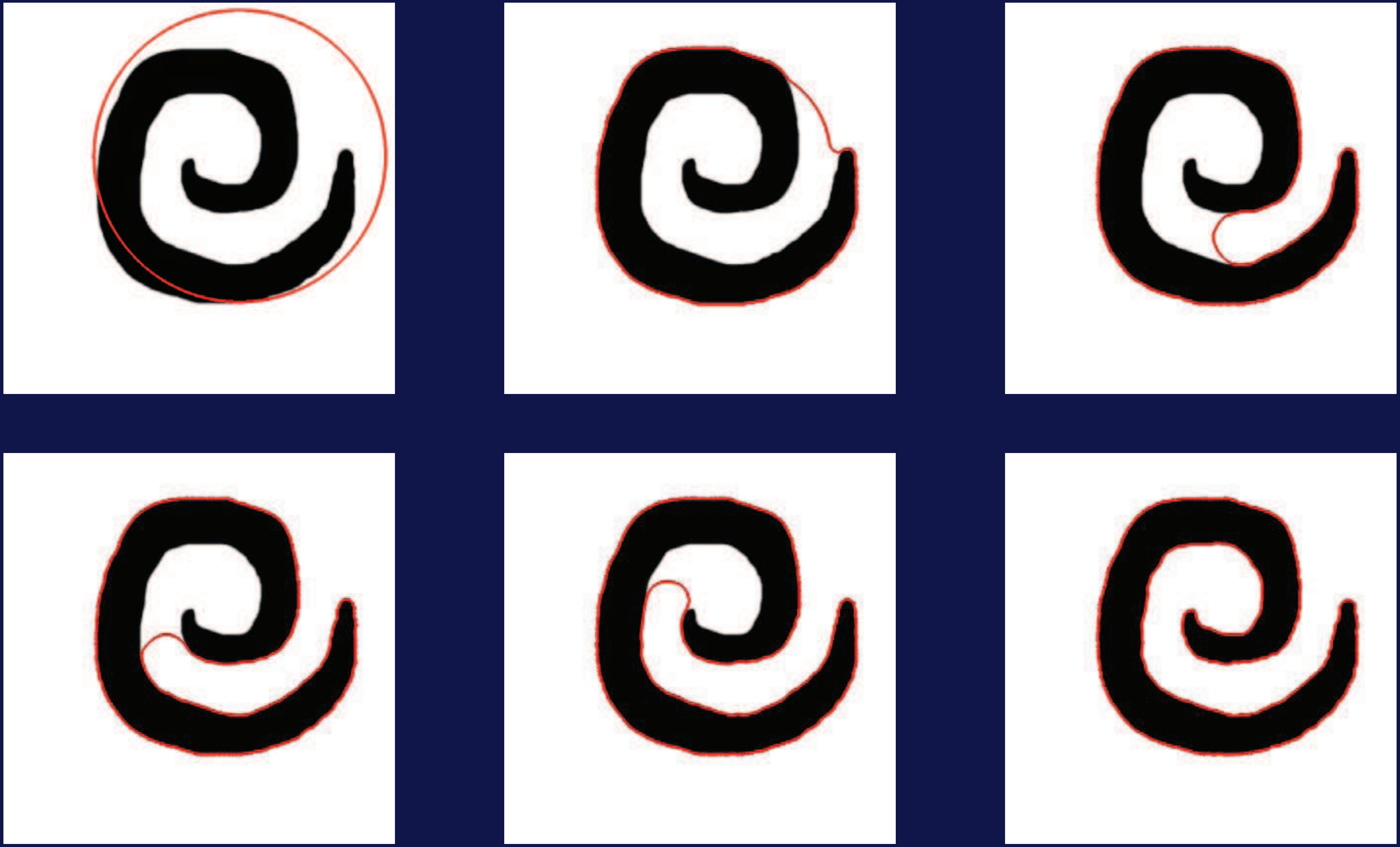
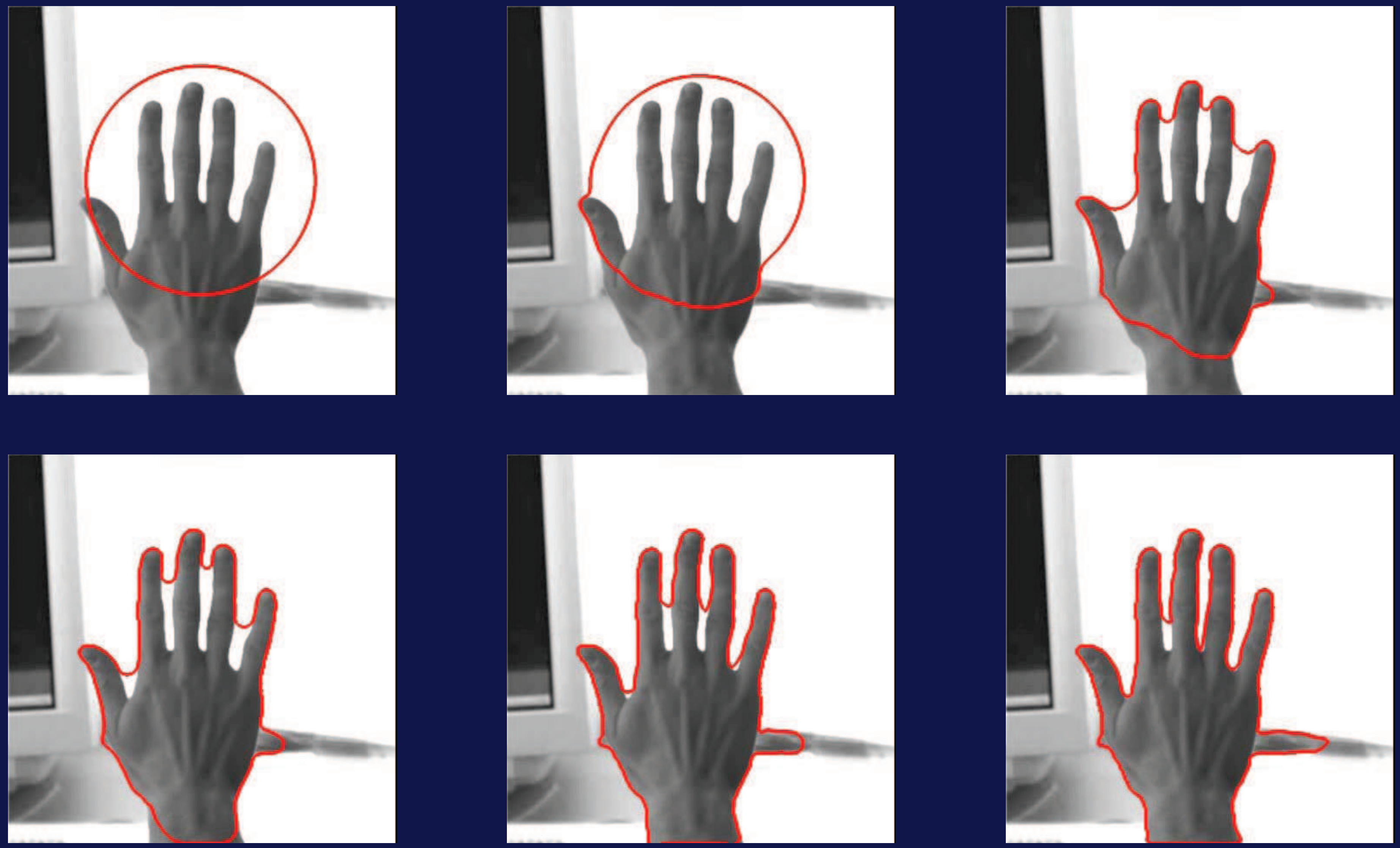
Evolution of Explicit Boundaries



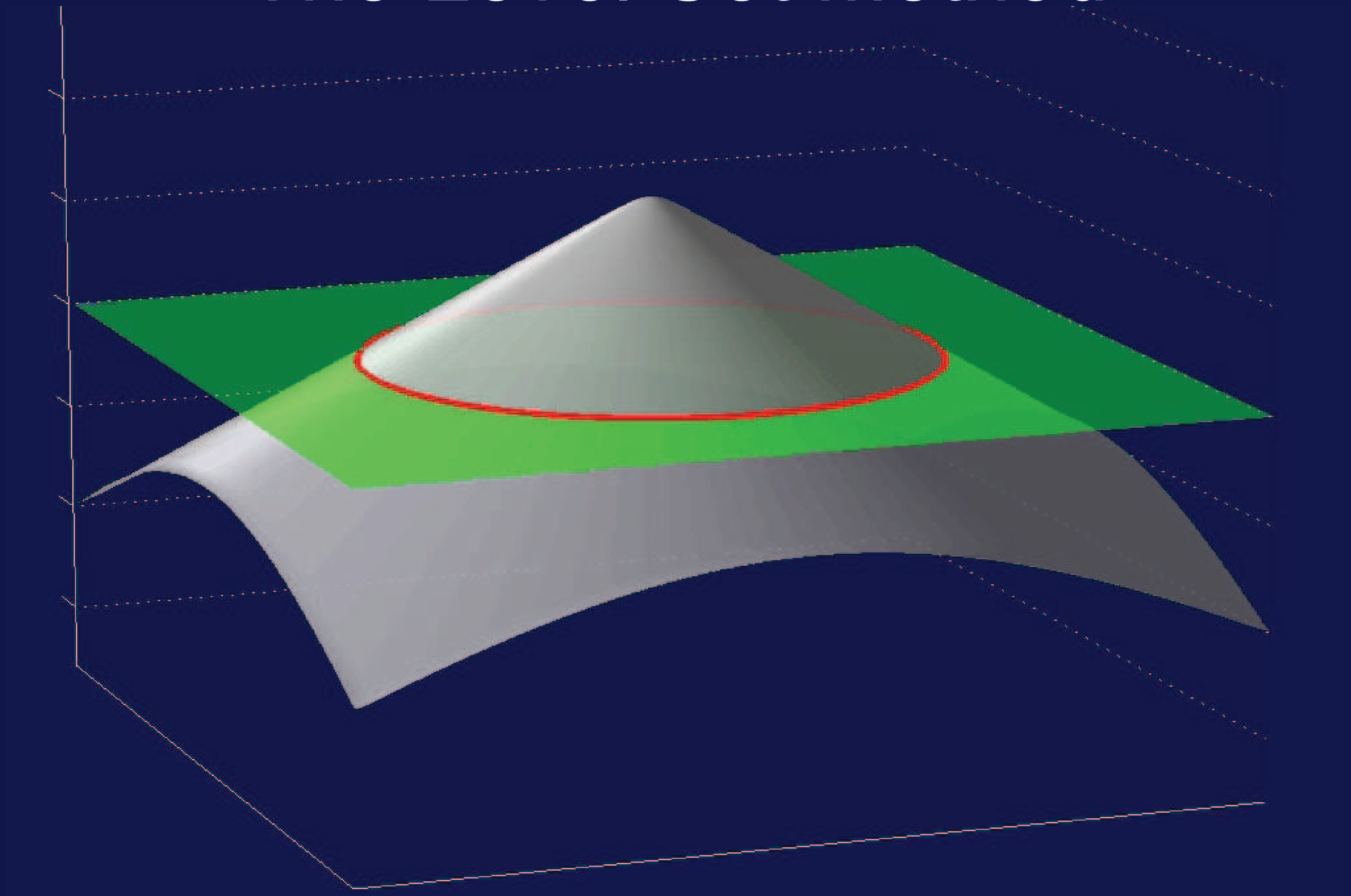
Cremers, Tischhäuser, Weickert, Schnörr, "Diffusion Snakes", IJCV '02

Evolution of Explicit Boundaries



Cremers, Tischhäuser, Weickert, Schnörr, “Diffusion Snakes”, IJCV '02

The Level Set Method



$$C = \{x \in \Omega \,|\, \phi(x) = 0\}, \quad \phi : \Omega \subset \mathbb{R}^n \to \mathbb{R}$$

Osher, Sethian, J. of Comp. Phys. ’88

Dervieux, Thomasset, ’79, ’81

The Level Set Method

Assume the interface evolves according to: $\frac{dC}{dt} = F\,\vec{n}$

At all times the interface is the zero level of ϕ :

$$\phi\big(C(t), t\big) = 0 \quad \forall\, t.$$

Then the total time derivative of must vanish:

$$0 = \frac{d}{dt}\,\phi\big(C(t), t\big) = \nabla\phi\,\frac{dC}{dt} + \partial_t\phi.$$

We obtain an evolution equation for ϕ : $\partial_t\phi = -\nabla\phi\,\frac{dC}{dt} = -\nabla\phi\,F\vec{n}.$

Using $\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$, we obtain the level set equation:

$$\boxed{\partial_t\phi = -F\,|\nabla\phi|.}$$

Image Segmentation: Edge-based

Kass, Witkin, Terzopoulos, “Snakes” ‘88:

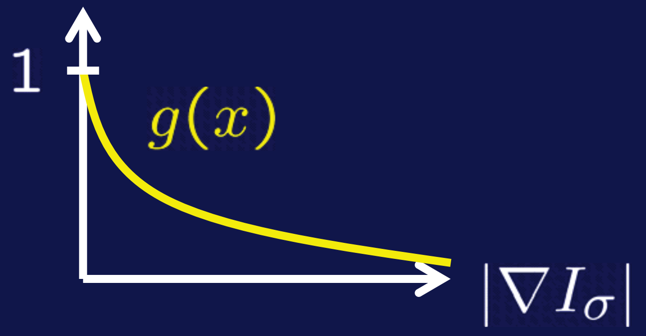
$$E(C) = -\underbrace{\int |\nabla I(C)|^2 ds}_{\text{external energy}} + \underbrace{\int \left\{ \nu_1 |C_s|^2 + \nu_2 |C_{ss}|^2 \right\} ds}_{\text{internal energy}}$$

Image $I : \Omega \to \mathbb{R}$, parametric contour $C : [0,1] \to \Omega$

Caselles et al.’93, Caselles et al.’95, Kichenassamy et al. ‘95:

$$E(C) = \int g(C(s))\, ds \qquad g(x) = \frac{1}{1 + |\nabla I_\sigma(x)|^2}$$

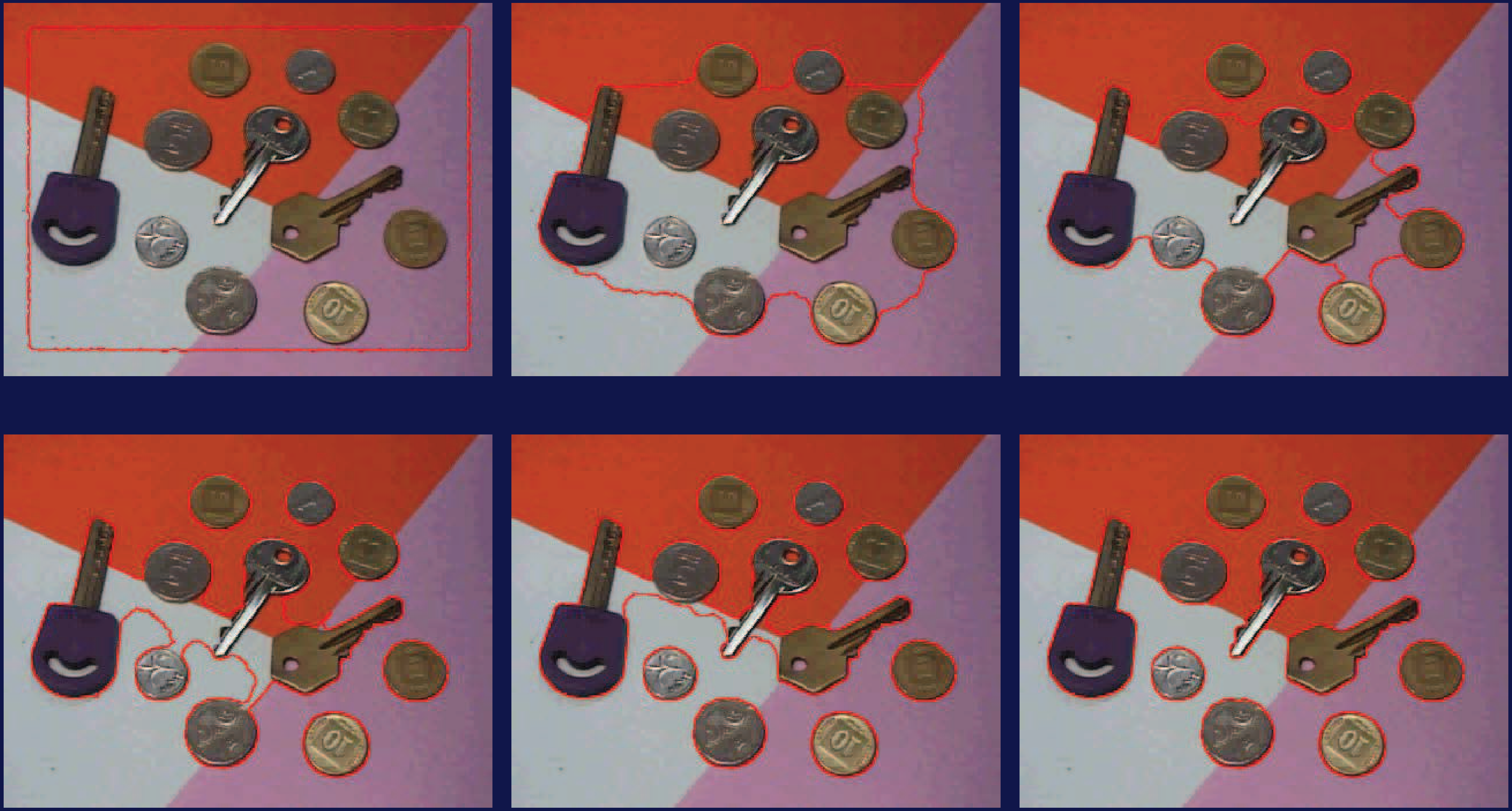
edge indicator function smoothed image



geodesic active contours:

$$\frac{\partial \phi}{\partial t} = |\nabla \phi| \operatorname{div} \left(g(x) \frac{\nabla \phi}{|\nabla \phi|} \right)$$

Image Segmentation: Edge-based



Goldenberg, Kimmel, Rivlin, Rudzsky, IEEE TIP ‘01

Image Segmentation: Region-based

$$E(u,K) = \int_{\Omega} (I-u)^2 dx + \lambda \int_{\Omega \setminus K} |\nabla u|^2 dx + \nu_o \mathcal{H}^1(K)$$

Mumford, Shah '85, '89
Blake, Zisserman '87

$\Omega \subset I\!R^2$ Image domain

$I : \Omega \rightarrow I\!R$ Input image

$u : \Omega \rightarrow I\!R$ Segmented image

$K \subset \Omega$ Discontinuity set

$\lambda \rightarrow \infty :$ piecewise constant model

$$E(u,K) = \sum_i \int_{R_i} \big(I(x) - u_i\big)^2 dx + \nu\,|K|$$

Mumford, Shah '89
spatially discrete: *Blake '83*

Image Segmentation: Region-based

$$E(u,K) = \sum_i \int_{R_i} \left(I(x) - u_i \right)^2 dx \; + \; \nu \, |K|$$

piecewise constant model

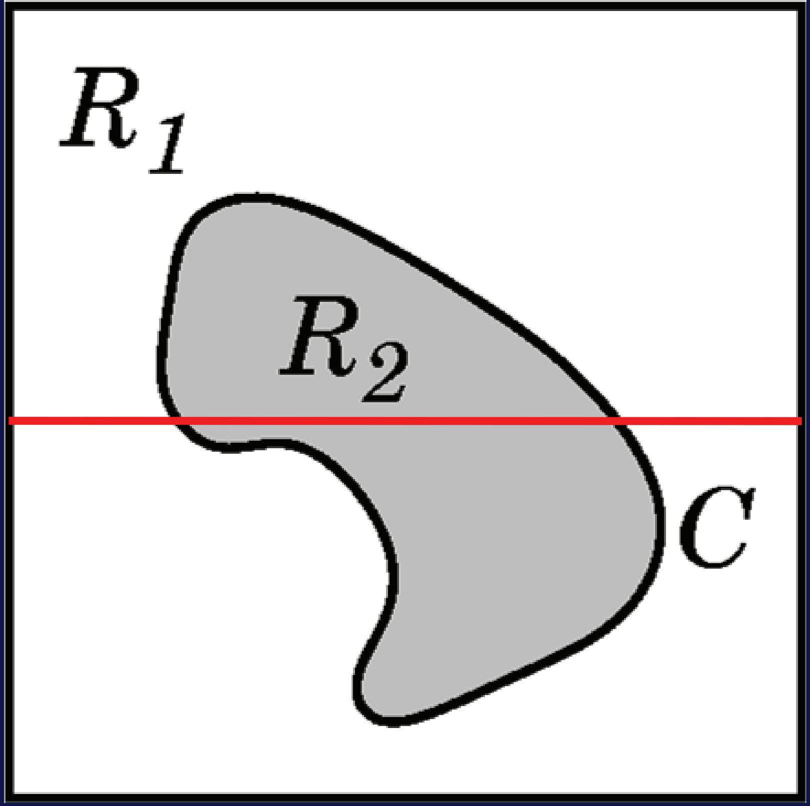
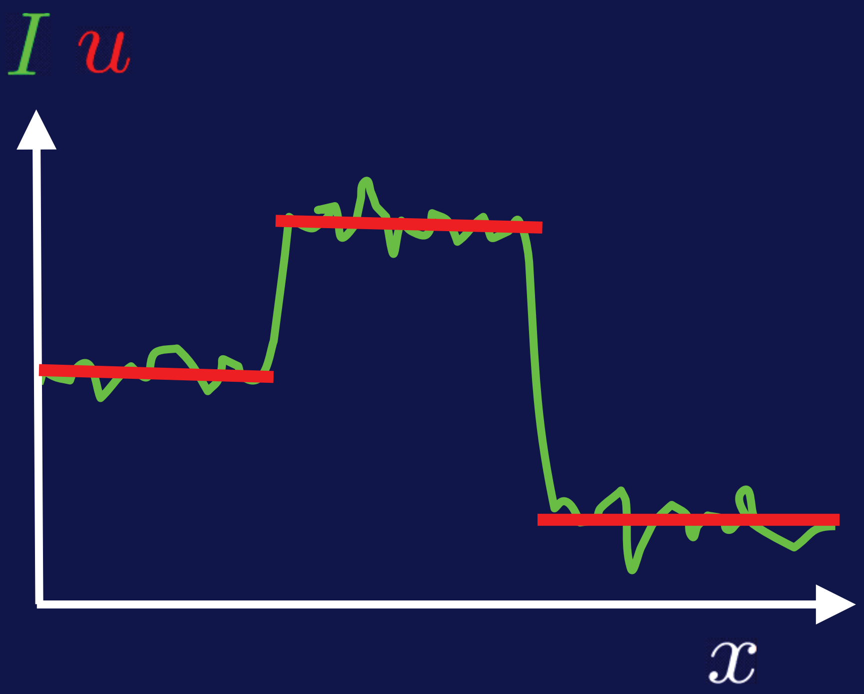


Image Segmentation: Region-based

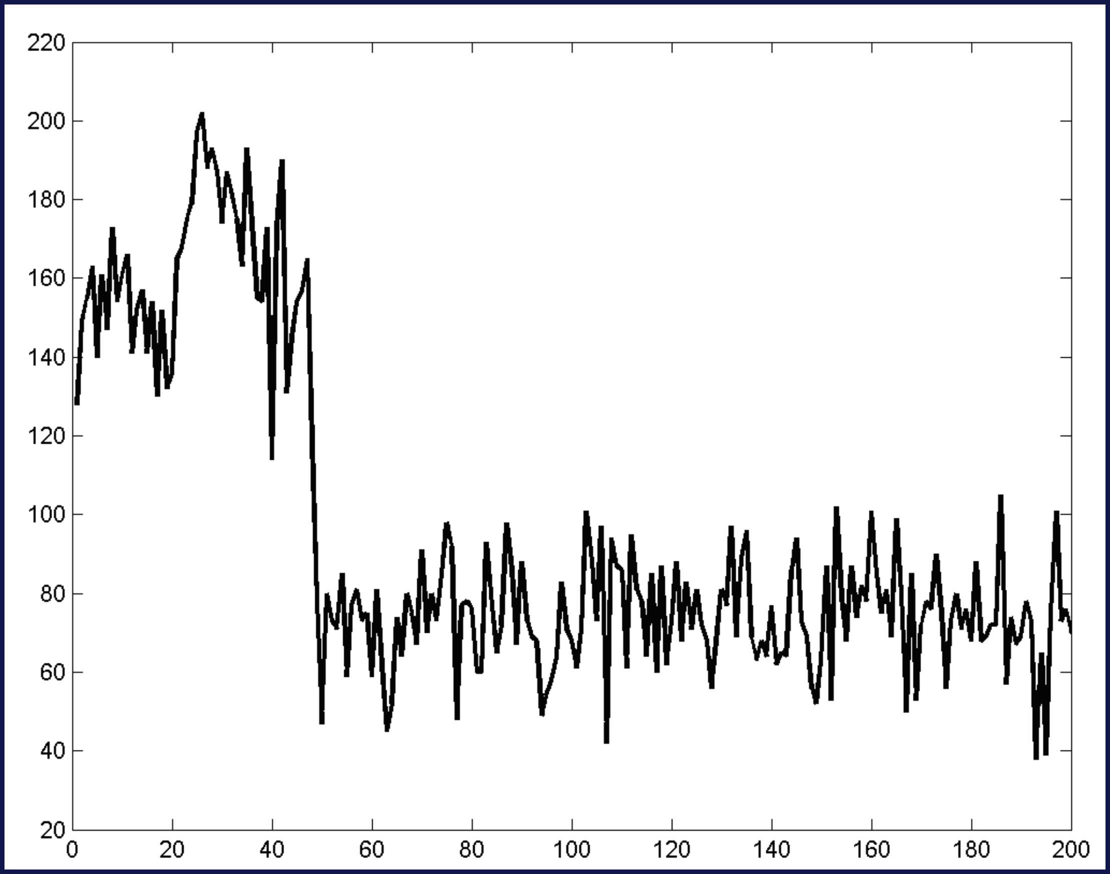
$$E(u,K) = \sum_i \int_{R_i} \left(I(x) - u_i\right)^2 dx \; + \; \nu \, |K|$$

piecewise constant model



Quantitative Comparison on 1-D

Input image

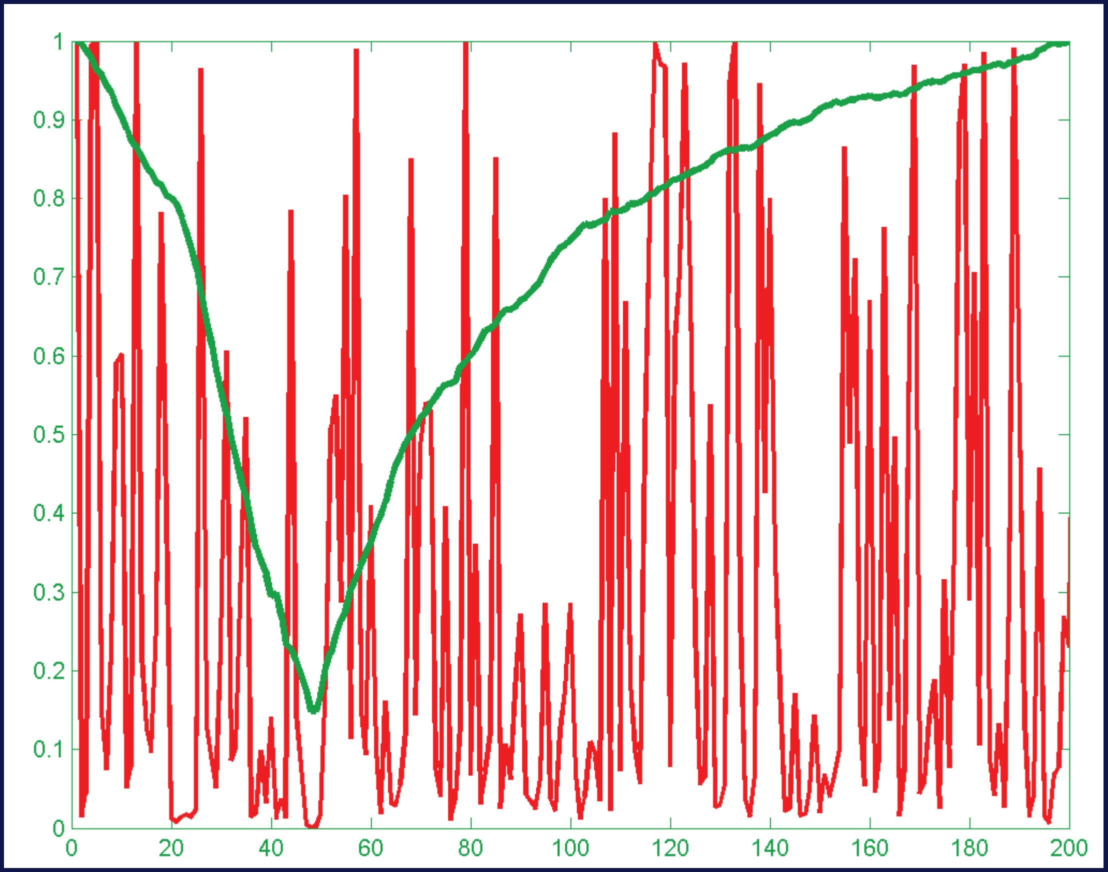


Intensity along 1D slice

Cremers, Rousson, Deriche, IJCV '06

Quantitative Comparison on 1-D

Input image

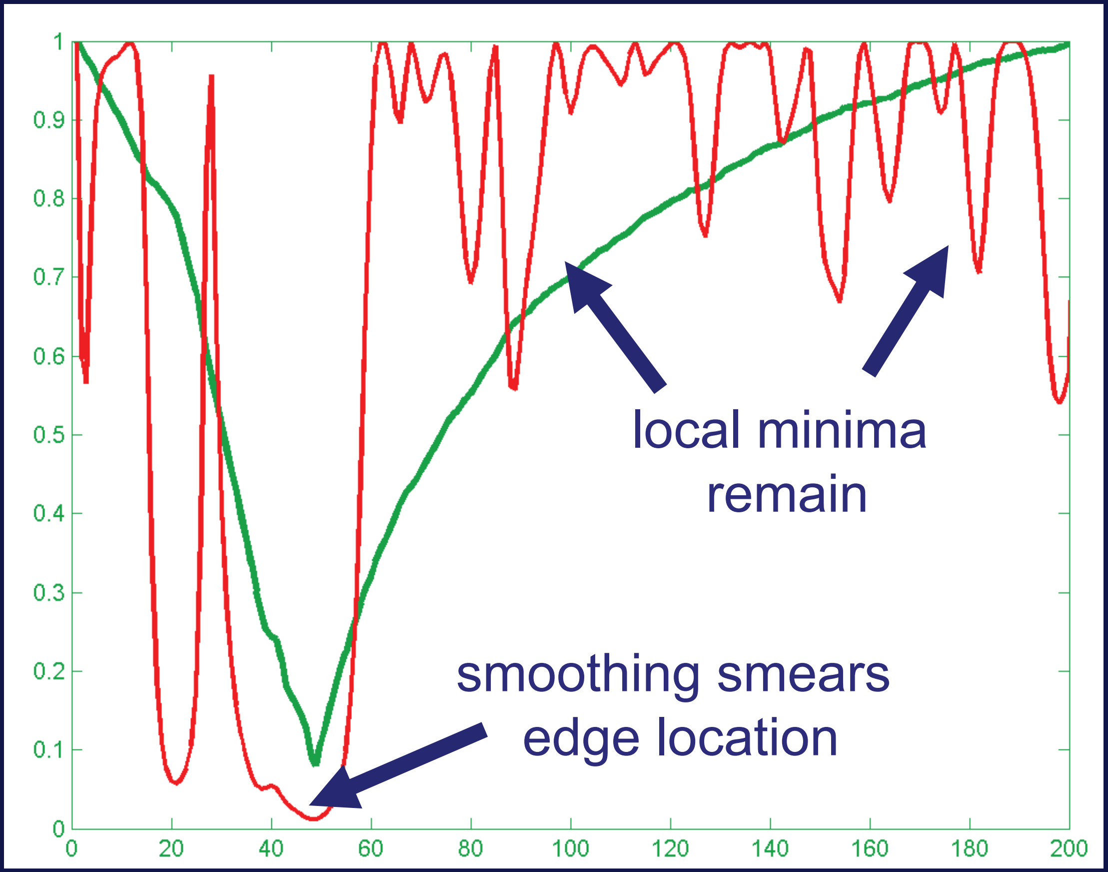


Energy of 1D segmentation for
edge-based and region-based energies

Cremers, Rousson, Deriche, IJCV ’06

Quantitative Comparison on 1-D

Input image



local minima remain

smoothing smears edge location

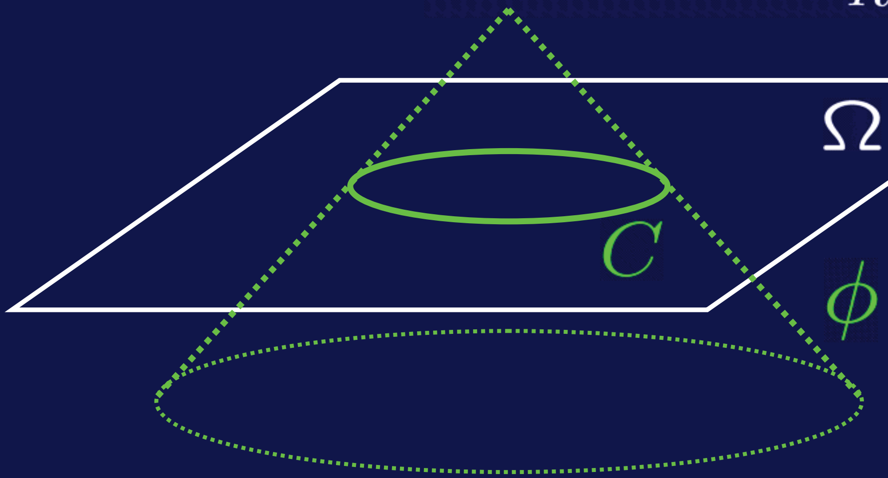
Energy for region-based and edge-based after smoothing

Cremers, Rousson, Deriche, IJCV '06

Level Set Formulation of Mumford-Shah

Chan, Vese ’99, Tsai et al. ’00

$$E(u,C) = \sum_i \int_{R_i} \left(I(x) - u_i\right)^2 dx + \nu\,|C|$$



$$H\phi \equiv H(\phi) = \begin{cases} 1, & \text{if } \phi > 0 \\ 0, & \text{else} \end{cases}$$

Use smoothed step function

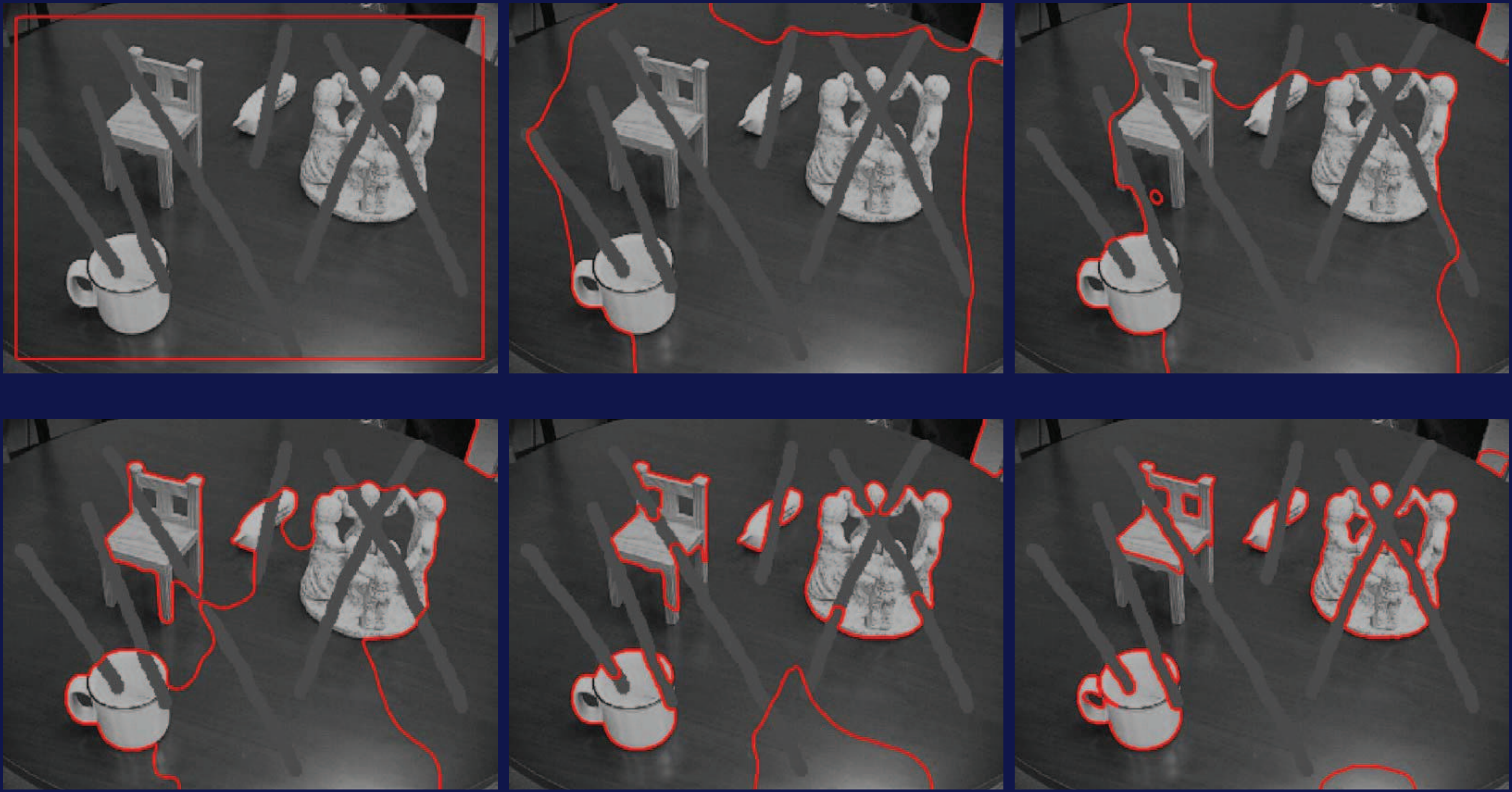
$$E(\phi,u) = \int_{\Omega} (I-u_1)^2 H\phi + (I-u_2)^2 \left(1 - H\phi\right) dx + \nu \int_{\Omega} |\nabla H\phi| dx$$

$$\boxed{\frac{\partial \phi}{\partial t} = -\frac{\partial E}{\partial \phi} = \delta(\phi) \left(\nu\,\text{div}\left(\frac{\nabla \phi}{|\nabla \phi|}\right) + (I-u_2)^2 - (I-u_1)^2 \right)}$$

Level Set Formulation of Mumford-Shah

Chan & Vese '99 , Implementation: D. Cremers

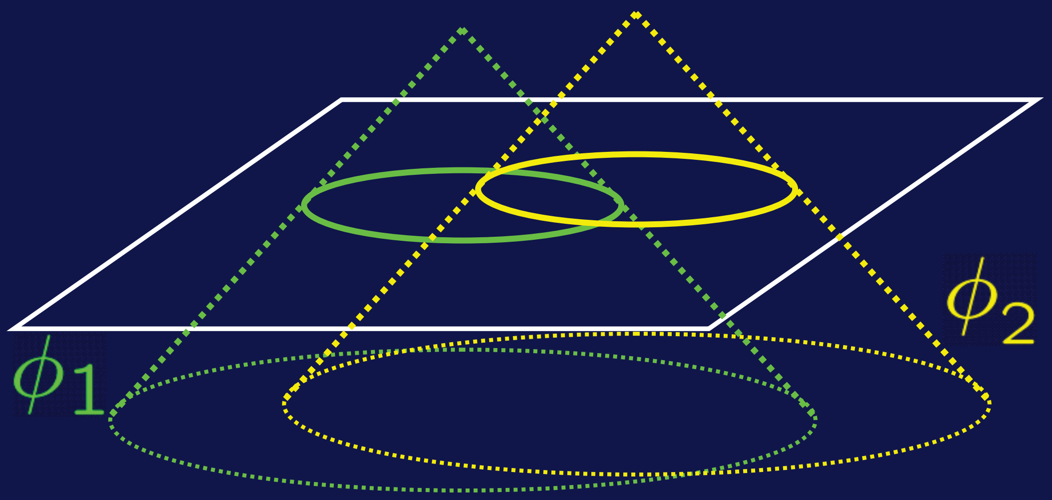
Level Set Formulation of Mumford-Shah



Chan & Vese ’99, Implementation: D. Cremers

Multiphase Level Set Formulation

Vese, Chan '02



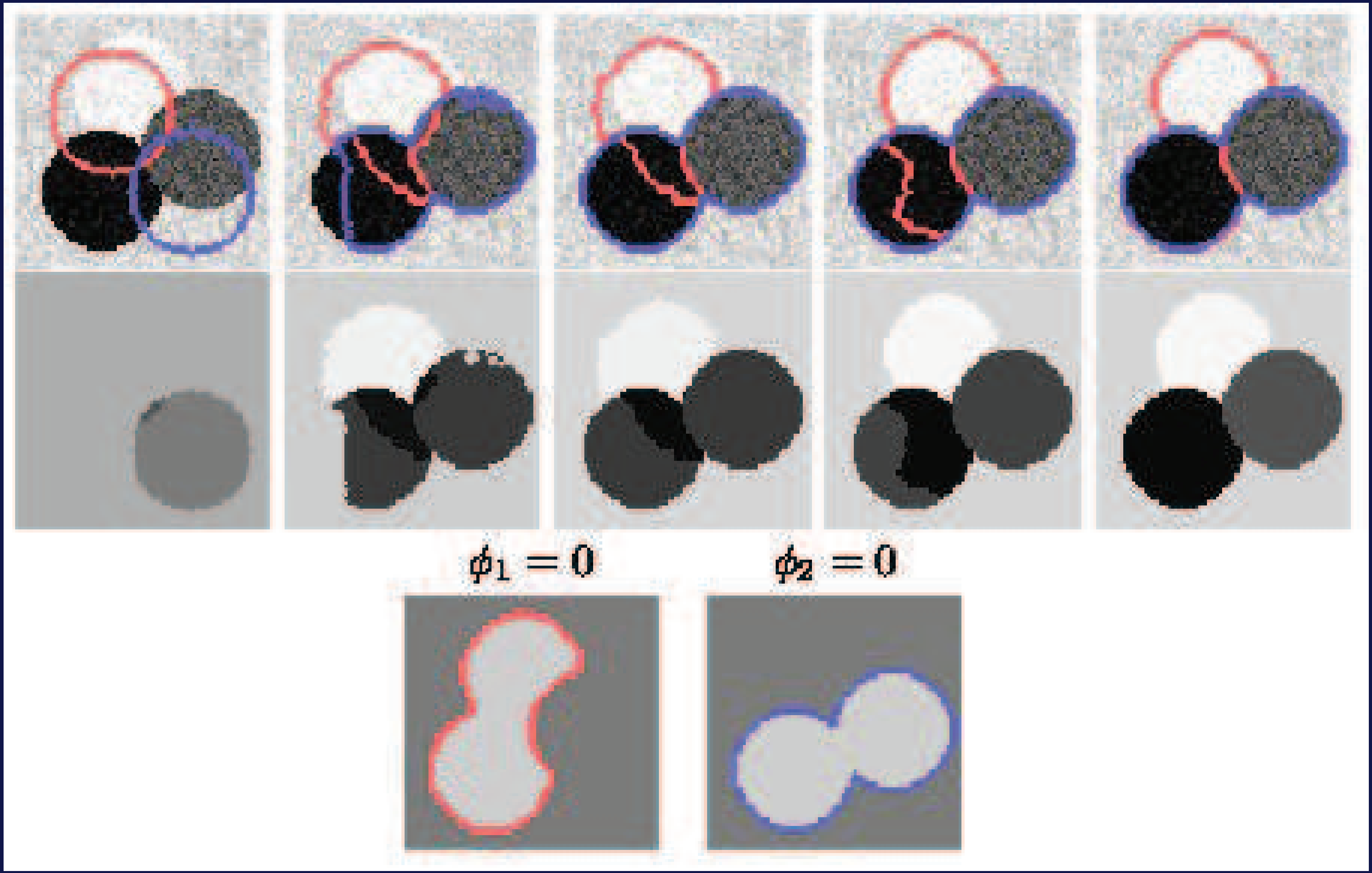
	$\phi_1 \geq 0$	$\phi_1 < 0$
$\phi_2 \geq 0$	Ω_1	Ω_2
$\phi_2 < 0$	Ω_3	Ω_4

$$E(\phi_1, \phi_2, u) = \int_\Omega (I - u_1)^2 H\phi_1 H\phi_2 + (I - u_2)^2 \left(1 - H\phi_1\right) H\phi_2 \, dx$$

$$+ \int_\Omega (I - u_3)^2 H\phi_1 \left(1 - H\phi_2\right) + (I - u_4)^2 \left(1 - H\phi_1\right)\left(1 - H\phi_2\right) dx$$

$$+ \nu \sum_i \int_\Omega |\nabla H\phi_i| dx \qquad \frac{\partial \vec{\phi}}{\partial t} = -\frac{dE}{d\vec{\phi}}$$

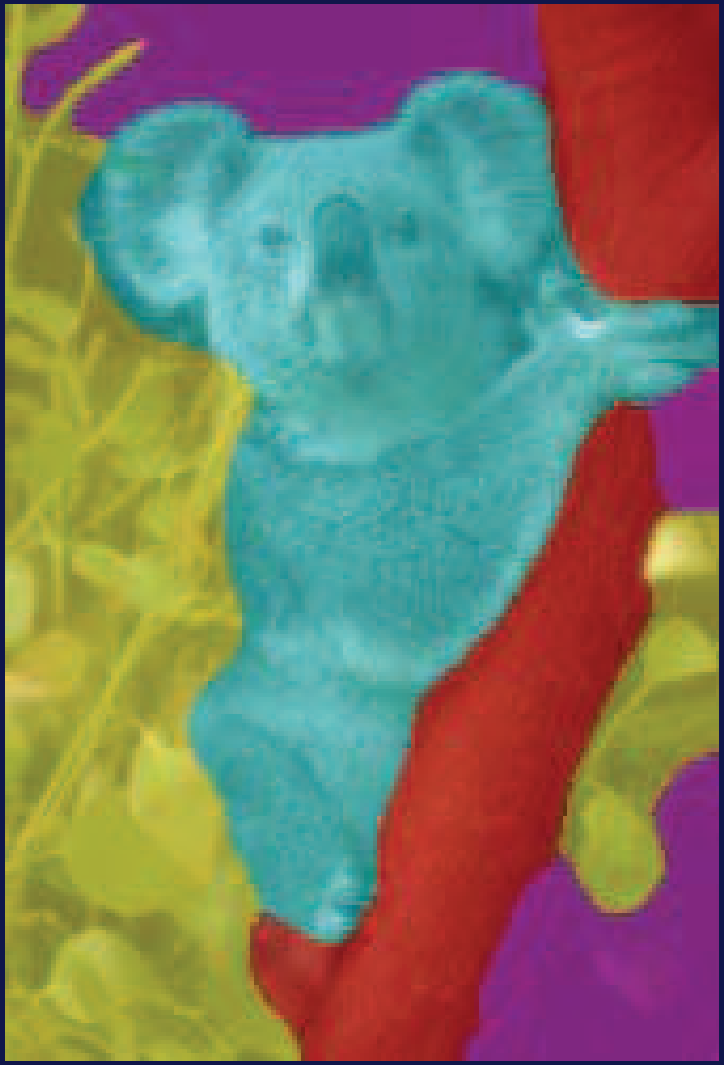
Multiphase Level Set Formulation



Chan, Vese '01

Efficient Multiphase Formulation

2-phase solution



multiphase solution

Brox, Weickert ’04, ’06

Efficient Multiphase Formulation

Brox, Weickert ’04, ’06