Nonlinear Least Squares

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EE103
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December 6, 2016

Location from range measurements

- 3-vector x is position in 3-D, which we will estimate
- *range* measurements give (noisy) distance to known locations

$$\rho_i = \|x - a_i\| + v_i, \quad i = 1, \ldots, m$$

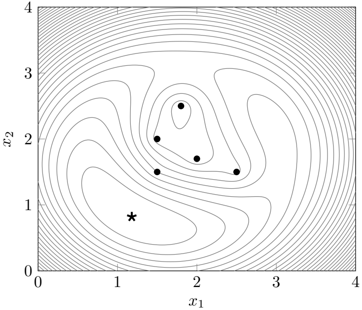
  - a_i are known locations
  - v_i are noises
- least squares location estimation: choose $\hat{x}$ that minimizes

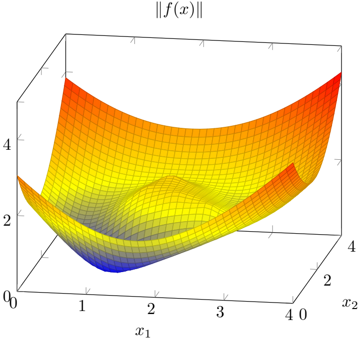
$$\sum_{i=1}^{m} \left(\|x - a_i\| - \rho_i\right)^2$$

- GPS works like this

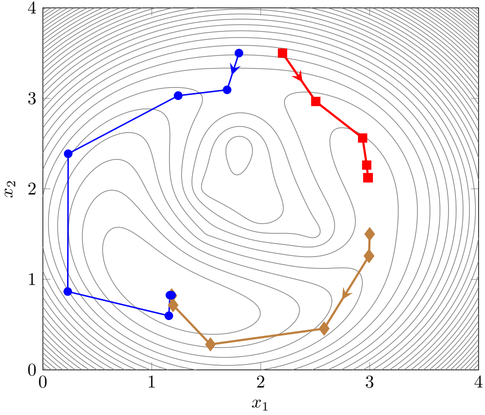
Example: Location from range measurements

range to 5 points (circles)





Levenberg-Marquardt from 3 initial points



Model fitting

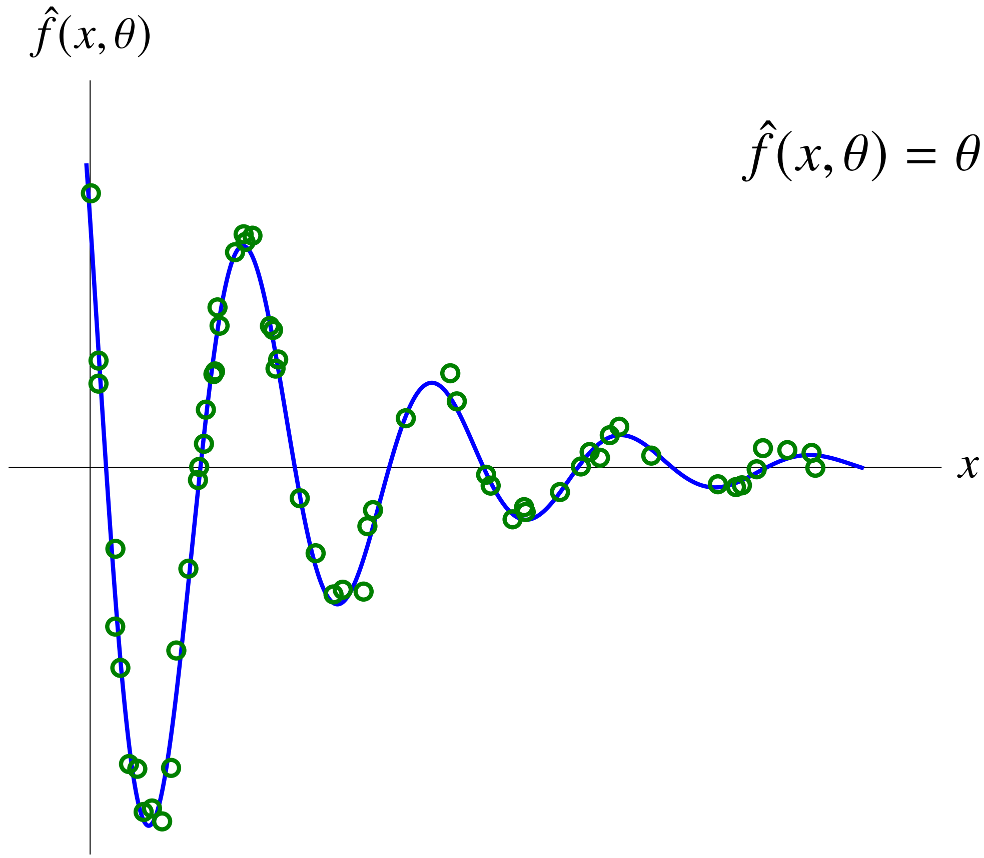
$$\text{minimize} \quad \sum_{i=1}^{N} (\hat{f}(x^{(i)}, \theta) - y^{(i)})^2$$

- model $\hat{f}(x, \theta)$ is parameterized by parameters $\theta_1, \ldots, \theta_p$
- $(x^{(1)}, y^{(1)}), \ldots, (x^{(N)}, y^{(N)})$ are data points
- the minimization is over the model parameters θ
- on page 9.9 we considered models that are linear in the parameters θ:

$$\hat{f}(x, \theta) = \theta_1 f_1(x) + \cdots + \theta_p f_p(x)$$

here we allow $\hat{f}(x, \theta)$ to be a nonlinear function of θ

Example



$$\hat{f}(x,\theta) = \theta_1 \exp(\theta_2 x) \cos(\theta_3 x + \theta_4)$$

a nonlinear least squares problem with four variables θ_1, θ_2, θ_3, θ_4:

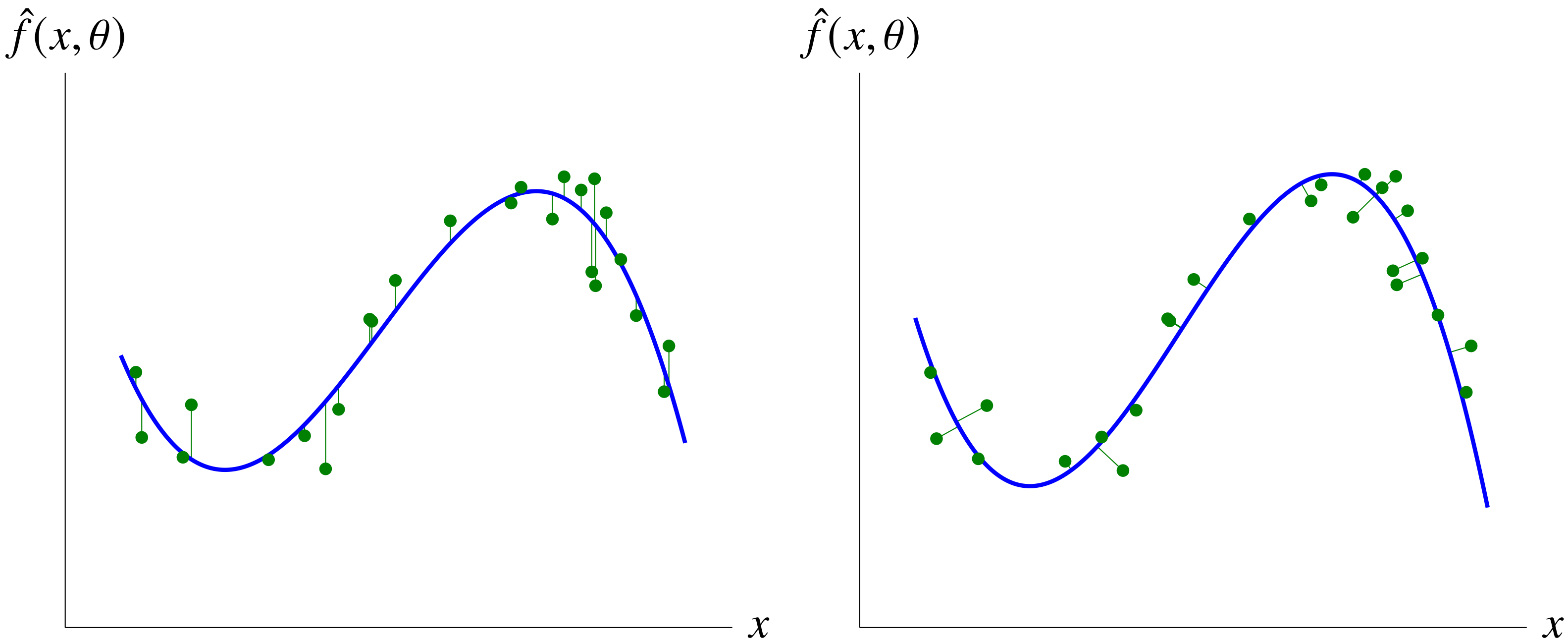
$$\text{minimize} \quad \sum_{i=1}^{N} \left(\theta_1 e^{\theta_2 x^{(i)}} \cos(\theta_3 x^{(i)} + \theta_4) - y^{(i)}\right)^2$$

Orthogonal distance regression

minimize the mean square distance of data points to graph of $\hat{f}(x, \theta)$

Example: orthogonal distance regression with cubic polynomial

$$\hat{f}(x, \theta) = \theta_1 + \theta_2 x + \theta_3 x^2 + \theta_4 x^3$$

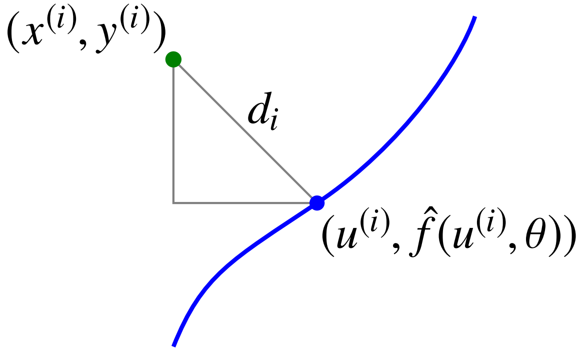


standard least squares fit   orthogonal distance fit

Nonlinear least squares formulation

$$\text{minimize} \quad \sum_{i=1}^{N} \left((\hat{f}(u^{(i)}, \theta) - y^{(i)})^2 + \|u^{(i)} - x^{(i)}\|^2 \right)$$

- optimization variables are model parameters θ and N points $u^{(i)}$
- ith term is squared distance of data point $(x^{(i)}, y^{(i)})$ to point $(u^{(i)}, \hat{f}(u^{(i)}, \theta))$



$$d_i^2 = (\hat{f}(u^{(i)}, \theta) - y^{(i)})^2 + \|u^{(i)} - x^{(i)}\|^2$$

- minimizing d_i^2 over $u^{(i)}$ gives squared distance of $(x^{(i)}, y^{(i)})$ to graph
- minimizing $\sum_i d_i^2$ over $u^{(1)}, \ldots, u^{(N)}$ and θ minimizes mean squared distance