Lecture 12: Bonuses – Symbolic Math

B0B17MTB, BE0B17MTB – MATLAB

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Outline

Mathematical Analysis

- Two different attitudes are distinguished:
  - symbolic math,
  - numerical math.
- Possible classification: analytical result, in principle, enables to get result in infinite number of decimals; numerical result is encumbered with numerical errors.
- There exist wide range of techniques in MATLAB (symbolical as well as numerical) and only selected topics are discussed in this lecture.

Function Handle and Anonymous Functions — Recap

- Enables indirect function invoking.
- Reference to the function is stored in variable hn:

```
hn = @function_name
```

- Function handle is a data type in MATLAB (see whos).
- Anonymous function (exists as variable without m-file):

```
as = @(args) function_name(args)
```

- Enables to invoke a function from locations where it is not visible to MATLAB:

```
>> fxy = @(x, y) x^2 + y^2 - 5
>> fxy(2, -2)

>> fcos = @(alpha) cos(alpha)
>> fcos(pi)
```

Polynomials I.

- Representation of polynomials in MATLAB

$$P = C_n x^n + C_{n-1} x^{n-1} + \cdots + C_1 x + C_0 = [C_n \quad C_{n-1} \quad \cdots \quad C_1 \quad C_0]\,. \tag{1}$$

- Roots of a polynomial: `roots`

```
>> x = roots([1 0 -1]);
>> x1 = x(1)
>> x2 = x(2)
```

- Polynomial evaluation: `polyval`

```
>> % x = 2
>> % p1 = 3*x^5 - 7*x^3 + 1/2*x^2 - 5
>> polyval([3 0 -7 1/2 0 -5], 2)
```

- Polynomial multiplication: `conv`

$A_1 = x - 1,$
$A_2 = x + 1,$
$A_1 A_2 = (x-1)(x+1) = x^2 - 1$

```
>> A1 = [1 -1]
>> A2 = [1 1]
>> conv(A1, A2)
% = [1 0 -1]
```

Symbolic Math

Polynomials II.

- Polynomial division: `deconv`:

$$\frac{x^2 - 1}{x + 1} = x - 1$$

```
>> deconv([1 0 -1], [1 1])
>> % = [1 -1]
```

- Some polynomial-related functions:
  - `residue`: residue of ratio of two polynomials,
  - `polyfit`: approximation of data with polynomial of order n,
  - `polyint`: polynomial integration,
  - `polyder`: polynomial derivative.

```
>> S = [1 1];
>> T = polyint(S) % = [0.5 1 0]
>> U = polyder(T) % = S = [1 1]
>> polyder(U)     % = 1
```

$$\int (x+1)\,\mathrm{d}x = \frac{1}{2}x^2 + x, \quad \frac{\mathrm{d}\left(\frac{1}{2}x^2 + x\right)}{\mathrm{d}x} = x + 1$$

Polynomials III.

- Polynomial multiplication (symbolic evaluation):

$$P_1 = A + Bx, \quad P_2 = 4x^2 + 2x - 4$$

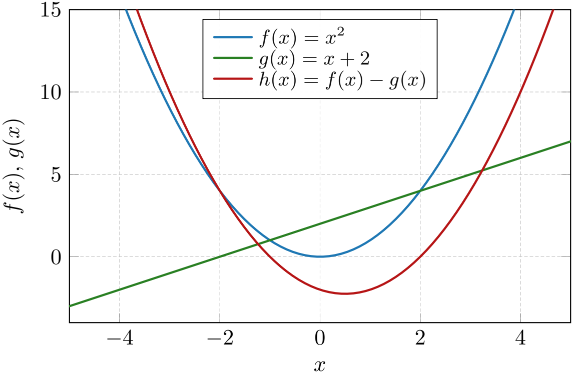
```
>> syms A B x
>> P1 = A + B*x;        % entering 1. polynomial
>> P2 = 4*x^2 + 2*x - 4; % 2. polynomial
>> P0 = P1*P2;          % multiplication
>> P  = expand(P0)       % expansion
```

- Note: function `expand` requires Symbolic Math Toolbox.

Symbolic Math

Intersection of Two Functions

- Two functions are given, $f(x) = x^2$ and $g(x) = x + 2$. Find $x : f(x) = g(x)$.
- Recast as to find $x : h(x) = 0$, where $h(x) = f(x) - g(x)$:



To find the solution:

```
>> syms x;
>> f = x^2;
>> g = x + 2;
>> x0 = solve(f - g) % = 2; -1
```

To depict the functions:

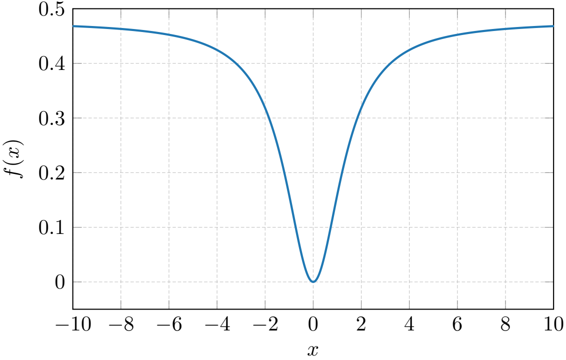
```
>> fplot(f);
>> hold on;
>> grid on;
>> fplot(g);
```

Symbolic Math

Limit and Derivative I.

- Find out function limit:

$$f(x) = \frac{3x^3}{2\pi x^3 + 4\pi x} = \frac{3}{2\pi}\left(\frac{x^2}{x^2+2}\right), \text{ L'Hôpital's rule: } \lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = \frac{3}{2\pi} \approx 0.4775$$




```
>> syms x real;
>> f = 3*x^3/(2*pi*x^3 + 4*pi*x)

>> lim1 = limit(f, x, -inf)
>> lim2 = limit(f, x, inf)

>> double(lim1) % = 0.4775
>> double(lim2) % = 0.4775

>> figure;
>> fplot(f);
>> grid on;
```

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Limit and Derivative I.

- Apply L'Hôspital's rule to previous function:
  - Function $f(x)$ contains 3rd power of x; carry out 3rd derivative (of numerator $f_1(x)$ and denominator $f_2(x)$ separately) in x.

$$f(x) = \frac{3x^3}{2\pi x^3 + 4\pi x}, \quad f_1(x) = 3x^3, \quad f_2(x) = 2\pi x^3 + 4\pi x$$

```
>> f1 = 3*x^3;
>> f2 = 2*pi*x^3 + 4*pi*x;
>> A1 = diff(f1,3)
>> A2 = diff(f2,3)
>> double(A1/A2) % = 0.4775
```

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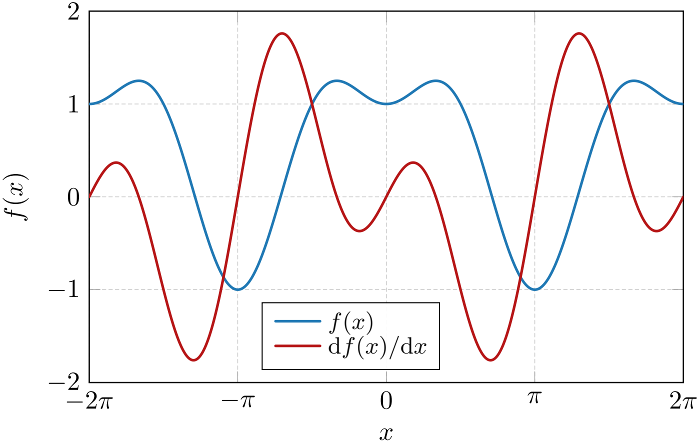
Limit and Derivative I

- Carry out derivative of the following function in $f(x) = \sin^2(x) + \cos(x)$.
  - Compare results and plot them.



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Limit and Derivative I

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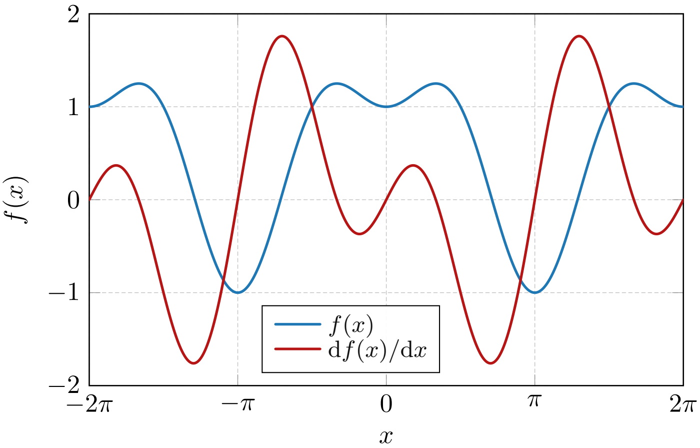


```
>> syms x;
>> f = sin(x)^2 + cos(x);
>> figure; fplot(f);
>> fd = diff(f);
>> figure; fplot(fd);
```

Integration I.

- Let's first symbolically carry out derivative of function $f(x) = \sin(x) + 2$.
- Save the second derivative of $f(x)$ and call it $g(x)$, compare the results.
- Now, integrate function $g(x)$ (once, twice), do we get the original function $f(x)$?
  - Ignore integration constants.

```
>> clear, clc;
>> x = sym('x');

>> f = sin(x) + 2
>> figure; fplot(f);

>> fd = diff(f)
>> figure; fplot(fd);

>> fdd = diff(f, 2)
>> figure; fplot(fdd);
```

```
>> g = fdd;
>> gi = int(g)
>> figure; fplot(gi);

>> gii = int(gi);
>> err = f - gii

>> figure;
>> subplot(1, 2, 1);
>> fplot(f);
>> subplot(1, 2, 2);
>> fplot(gii);
```

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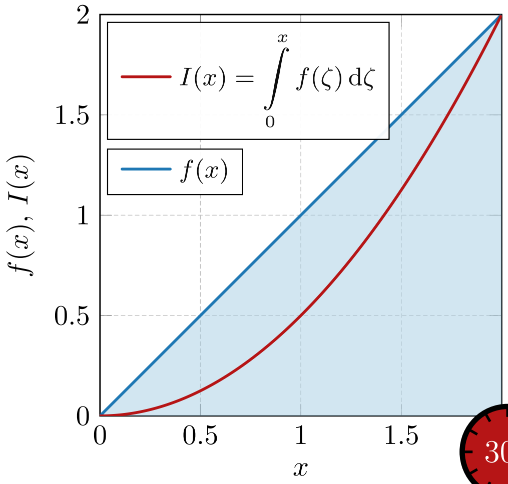
Integration I.

- Consider a function $f(x) = x$,
  - calculate indefinite integral of $f(x)$,
  - calculate definite integral for $x \in [0, 2]$, use, *e.g.*, function `int` or `integral`,

$$I = \int_0^2 f(x)\,\mathrm{d}x = \left[\frac{x^2}{2}\right]_0^2 = 2.$$

```
>> fill([0 2 2], [0 0 2], 'y')
```



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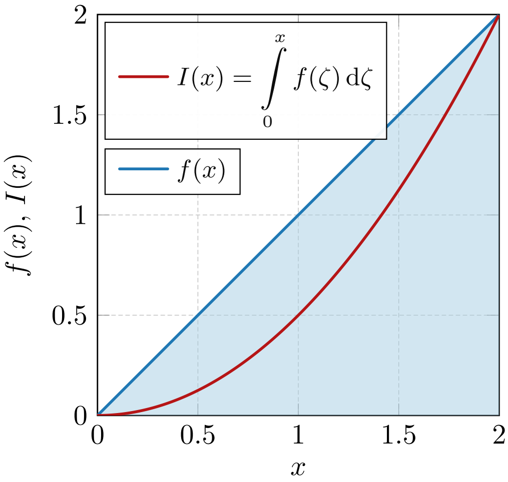
$$I = \int_0^2 f(x)\,\mathrm{d}x = \left[\frac{x^2}{2}\right]_0^2 = 2.$$

```
>> fill([0 2 2], [0 0 2], 'y')
```

```
>> syms x;
>> f = x;
>> g = int(f);
>> int(f, x, 0, 2)            % = 2
>> polyarea([0 2 2], [0 0 2]) % = 2
>> I = integral(@(x) x, 0, 2) % = 2
```

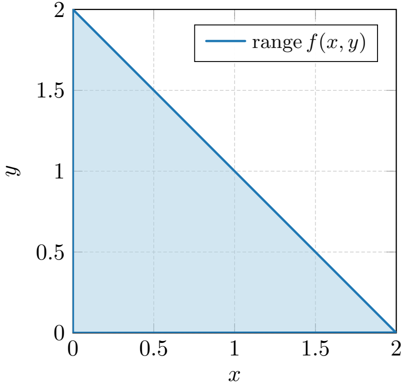


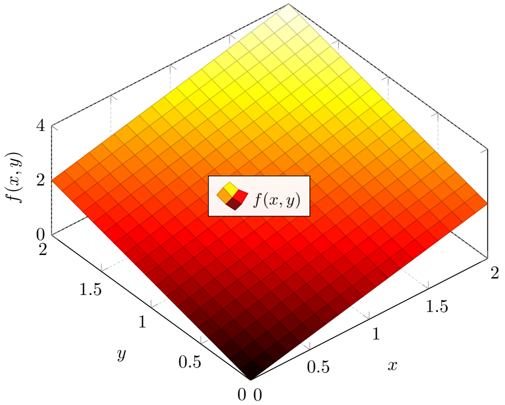
Numerical Integration I.

- Numerical approach is used whenever the closed-form (analytical) solution is not known which happens quite often in technical sciences (almost always).
- It is possible to use various numerical integration methods, see literature.
- Alternatively, MATLAB functions can be utilized:
  - `integral`, `integral2`, `integral3`, `quadgk`, and others,
  - define function to be integrated (write your own function or use function handle).

Numerical Integration II.

- Solve the following integral $I = \int\limits_0^2 \int\limits_0^{2-x} (x+y)\, \mathrm{d}y\, \mathrm{d}x = \cdots = \int\limits_0^2 \left(2 - \frac{x^2}{2}\right) \mathrm{d}x \approx 2.666$





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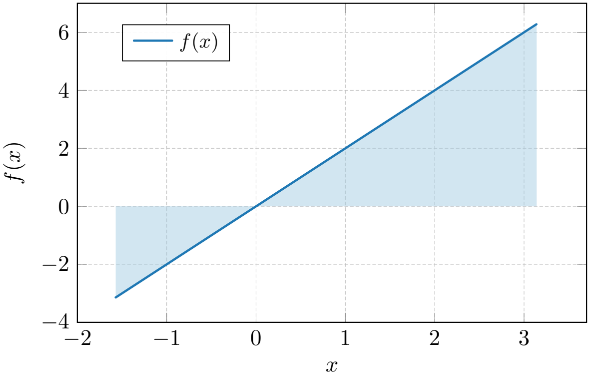
Numerical Integration III.

```
>> clear, clc;
% solution:
>> f = @(x, y) x + y
>> ymax = @(x) 2 - x
>> integral2(f, 0, 2, 0, ymax)

% plotting
>> t = 0:1/10:2
>> [x, y] = meshgrid(t);
>> z = x + y;
>> figure('color', 'w');
>> mesh(x, y, z);
```

Numerical Integration IV.

- It is possible to work with external scripts as well; *i.e.*, having "complex" expression that we don't want to process as handle: $I = \int\limits_{-\pi/2}^{\pi} f(x)\,\mathrm{d}x = \cdots = \frac{3}{4}\pi^2$.




```
function fx = myIntFcn(x)
% function to calculate integral:
% int{2*x}

c  = 2;
fx = c*x;
```

```
>> integral(@myIntFcn, -pi/2, pi)
```

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Closing Notes

- General problem of derivative (it is not possible to approach zero), *i.e.*,

$$\lim_{\Delta x \to 0} \frac{f\left(x + \Delta x\right) - f\left(x\right)}{\Delta x}$$

- Various sophisticated numerical methods of various complexity are used.
- Web pages to solve this problem in a complex way:
  - http://www.matrixlab-examples.com/derivative.html
- In the case there is a lot of symbolic calculations or when approaching MATLAB limits, try another mathematical tool (*e.g.*, Mathematica, Maple).
- Nevertheless, MATLAB is a perfect choice for numerical computing (although both Mathematica's symbolic and numerical kernels are excellent).

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Questions?

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