Homework No. 09

The lecture briefly described the evaluation of fundamental bounds based on the method of moments applied to electric field integral equation. In this homework, the evaluation of fundamental bounds is practiced on an example of maximum gain of a cylindrical radiator.

Setup: Assume, similarly to the last homework, a dipole placed along the z-axis ($z \in (-L/2, L/2)$) which has a form of a thin-wall highly conducting tube (not perfectly conducting as in the last homework). No cap is assumed at the end of the dipole. Assume that the dipole is described by an equivalent surface current density

$$\boldsymbol{K}_{\mathrm{e}} = I(z) \frac{\mathbf{z}_0}{2\pi a}, \; z \in (-L/2, L/2), \; \rho = a, \tag{1}$$

which is expanded into a set of basis functions as

$$\boldsymbol{K}_{\mathrm{e}} = \sum_{n=1}^{N} I_n \boldsymbol{\psi}_n, \tag{2}$$

with

$$\boldsymbol{\psi}_n = \frac{\mathbf{z}_0}{2\pi a} \sin\left(n\pi\left(\frac{z}{L} - \frac{1}{2}\right)\right), \; z \in (-L/2, L/2), \; \rho = a. \tag{3}$$

Electrodynamics of this system is described by impedance matrix $\mathbf{Z} = \mathbf{Z}_0 + \mathbf{Z}_\rho$, where the vacuum part of impedance matrix $\mathbf{Z}_0$ has been discussed in the last homework. Following the lectures, the material part of the impedance matrix is given by

$$Z_{\rho,mn} = \langle \boldsymbol{\psi}_m, Z_{\mathrm{s}} \boldsymbol{\psi}_n \rangle, \tag{4}$$

where surface impedance Z_{s} comes from a limiting process

$$\sigma^{-1} \boldsymbol{J}_{\mathrm{e}} \to Z_{\mathrm{s}} \boldsymbol{K}_{\mathrm{e}}, \tag{5}$$

where assumption of very high conductivity $\sigma \gg 1$ has been made.

An electric far field $\boldsymbol{F}(\theta, \varphi)$ corresponding to current density (2) can be evacuated as

$$\begin{bmatrix} \boldsymbol{\theta}_0 \cdot \boldsymbol{F} \\ \boldsymbol{\varphi}_0 \cdot \boldsymbol{F} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_\theta \\ \mathbf{F}_\varphi \end{bmatrix} \mathbf{I}, \tag{6}$$

where $\mathbf{F}_{(\theta/\varphi)}$ is a row vector given by

$$F_{(\theta/\varphi),n} = -\frac{\mathrm{j}\eta_0 k_0}{4\pi} \int\limits_S (\boldsymbol{\theta}_0 / \boldsymbol{\varphi}_0) \cdot \boldsymbol{\psi}_n(\boldsymbol{r}') \, \mathrm{e}^{\mathrm{j}k_0 \boldsymbol{r}_0 \cdot \boldsymbol{r}'} \mathrm{d}S' =$$

$$= -\frac{\mathrm{j}\eta_0 k_0 a}{2} \mathrm{J}_0(k_0 a \sin\theta) \int\limits_{-L/2}^{L/2} (\boldsymbol{\theta}_0 / \boldsymbol{\varphi}_0) \cdot \boldsymbol{\psi}_n(z') \, \mathrm{e}^{\mathrm{j}k_0 z' \cos\theta} \mathrm{d}z', \tag{7}$$

where J_0 is Bessel's function of order zero, η_0 is free-space impedance and k_0 is free-space wavenumer. Due to the vector orientation of basis functions, the azimuthal component of the far-field vector vanishes, i.e., $\mathbf{F}_\varphi = \mathbf{0}$.

Following the lectures, the gain G in direction $\boldsymbol{r}_0$ generated by this setup reads

$$G\left(\boldsymbol{r}_0\right) = \frac{4\pi U\left(\boldsymbol{r}_0\right)}{P_{\mathrm{rad}} + P_{\mathrm{lost}}}, \tag{8}$$

where radiation intensity U is evaluated as

$$U = \frac{1}{2}\mathbf{I}^{\mathrm{H}}\mathbf{U}\mathbf{I}, \tag{9}$$

with

$$\mathbf{U} = \frac{1}{\eta}\left(\mathbf{F}_{\theta}^{\mathrm{H}}\mathbf{F}_{\theta} + \mathbf{F}_{\varphi}^{\mathrm{H}}\mathbf{F}_{\varphi}\right) \tag{10}$$

and where

$$P_{\mathrm{rad}} + P_{\mathrm{lost}} = \frac{1}{2}\mathbf{I}^{\mathrm{H}}\mathrm{Re}\left\{\mathbf{Z}_0 + \mathbf{Z}_{\rho}\right\}\mathbf{I} \tag{11}$$

is the sum of radiated and lost power.

Task No. 1: Assume that the dipole described above is fed by a ring of magnetic current as it was in homework No. 06. Evaluate the gain of this system as a function of angle θ. For numerical evaluation use $kL \approx 0.9\pi$, $L/a = 50$. To evaluate surface impedance Z_{s}, use a conducting-half-space model [1, Chap. 8.1], which gives

$$Z_{\mathrm{s}} = \frac{(1 + \mathrm{j})}{\sigma\delta}, \tag{12}$$

where δ is penetration depth into the conductor. Assume copper at frequency $f = 1\,\mathrm{GHz}$.

In contrast to homework No. 06, here we assume that there is no ground plane and that a physical dipole is actually fed by the ring of magnetic current.

Task No. 2: Using prescription

$$G\left(\boldsymbol{r}_0\right) = 4\pi\frac{\mathbf{I}^{\mathrm{H}}\mathbf{U}\left(\boldsymbol{r}_0\right)\mathbf{I}}{\mathbf{I}^{\mathrm{H}}\mathrm{Re}\left\{\mathbf{Z}_0 + \mathbf{Z}_{\rho}\right\}\mathbf{I}}, \tag{13}$$

find optimal current $\mathbf{I}_{\mathrm{opt}}$ giving the highest achievable gain at angle $\theta = \pi/2$. Compare this optimal current with the realized current from the previous task. Do the same comparison also for optimal and realized radiation patterns.

References

[1] J. D. Jackson, *Classical Electrodynamics*. Wiley, 3 ed., 1998.