Question 1.

Let X contain all real numbers from $[0;1]$ which can be represented using 256 bits. Let $\mathcal{H} = X$, and let the decision be given by an $H \in \mathcal{H}$ as

$$h(x) = 1 \text{ iff } x > H$$

Determine an m such that with probability at least 0.9, $\text{err}(h) < 0.1$, where h is an arbitrary hypothesis from $\mathcal{H}$ consistent with m i.i.d. examples from X. Estimate it

1. without using any upper bounds seen in the lecture
2. using the upper bound with $\ln |\mathcal{H}|$
3. using the upper bound with $\text{VC}(\mathcal{H})$

Answer:

We have

$$\begin{aligned} \epsilon &= 0.1 \\ \delta &= 1 - 0.9 = 0.1 \\ |\mathcal{H}| &= 2^{256} \end{aligned}$$

1. For a fixed h, the probability that it is "bad" ($\text{err}(h) > \epsilon$) and still consistent with m i.i.d. observations is at most $(1-\epsilon)^m = 0.9^m$.

   For an arbitrary $h \in \mathcal{H}$, we can bound the probability of at least one of them being "bad" by

   $$\sum_{h \in \mathcal{H}} (1-\epsilon)^m = |\mathcal{H}|(1-\epsilon)^m = 2^{256} 0.9^m$$

   We want this probability to be smaller than δ:

   $$\begin{aligned} |\mathcal{H}|(1-\epsilon)^m &< \delta \\ m &\geq \log_{1-\epsilon} \frac{\delta}{|\mathcal{H}|} \end{aligned}$$

   i.e.,

   $$m > \log_{0.9} \frac{0.1}{2^{256}} \approx 1707 \text{ examples (smallest such } m) \tag{1}$$

2. $$\begin{aligned} m &> \frac{1}{\epsilon} \ln \frac{|\mathcal{H}|}{\delta} \\ m &> \frac{1}{0.1} \ln \frac{2^{256}}{0.1} \approx 1798 \text{ examples (smallest such } m) \end{aligned}$$

   which is slightly greater than (1) because the upper bound $(1-\epsilon)^m < e^{-\epsilon m} (\epsilon > 0)$ is used in the derivation of the formula.

3. $\text{VC}(\mathcal{H}) = 1$ because a single number from X can evidently be shattered (classified positively or negatively by hypotheses from $\mathcal{H}$) but two different numbers from X cannot be shattered: the smaller cannot be made positive while the larger is negative.

   $$m > \frac{8}{\epsilon} \left(\text{VC}(\mathcal{H}) \cdot \ln \frac{16}{\epsilon} + \ln \frac{2}{\delta} \right) \approx 646 \text{ examples}$$