GVG Lab-12 Solution

Task 1. *Let us have a fundamental matrix*

$$\mathtt{F} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which of the following couples of points are projections of a single point in space?

1. $\vec{u}_{1\alpha_1} = [1,1]^\top$, $\vec{u}_{2\alpha_2} = [1,1]^\top$
2. $\vec{u}_{1\alpha_1} = [1,0]^\top$, $\vec{u}_{2\alpha_2} = [0,1]^\top$
3. $\vec{u}_{1\alpha_1} = [0,0]^\top$, $\vec{u}_{2\alpha_2} = [1,0]^\top$

Justify.

Solution:

Remark. *Two points x_1 and x_2 are projections of a world point X* ***if and only if***

$$\vec{x}_{2\beta_2}^\top \mathtt{F} \vec{x}_{1\beta_1} = 0 \text{ and } x_1 \neq e_1, x_2 \neq e_2 \tag{1}$$

or

$$x_1 = e_1, x_2 = e_2 \tag{2}$$

where e_1 and e_2 are the epipoles in the images. To understand the above statement imagine what happens if we take $x_1 = e_1$ and $x_2 \neq e_2$. Then $\vec{x}_{2\beta_2}^\top \mathtt{F} \vec{x}_{1\beta_1} = 0$ since $\mathtt{F}\vec{x}_{1\beta_1} = \mathtt{F}\vec{e}_{1\beta_1} = \mathbf{0}$, however the fact that $x_1 = e_1$ means that a world point is located on the baseline and thus its projection to the second image x_2 must be also the second epipole e_2. Thus, by verifying that $\vec{x}_{2\beta_2}^\top \mathtt{F} \vec{x}_{1\beta_1} = 0$ we can say that x_1 and x_2 are projections of a world point X only if we know that x_1 and x_2 are not both epipoles or if they are.

We first compute the epipoles in both cameras. We know that

$$\mathtt{F}\vec{e}_{1\beta_1} = \mathbf{0}, \quad \mathtt{F}^\top \vec{e}_{2\beta_2} = \mathbf{0}$$

Hence we need to compute kernels of $\mathtt{F}$ and $\mathtt{F}^\top$. We do it by Gaussian elimination:

$$\mathtt{F} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \ker \mathtt{F} = \left\{ \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\} = \left\langle \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{e}_{1\beta_1}} \right\rangle$$

$$\mathtt{F}^\top = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \ker \mathtt{F}^\top = \left\{ \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\} = \left\langle \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{\vec{e}_{2\beta_2}} \right\rangle$$

1. $\vec{u}_{1\alpha_1} = [1,1]^\top$, $\vec{u}_{2\alpha_2} = [1,1]^\top$. We verify the epipolar constraint

$$\vec{x}_{2\beta_2}^\top \mathtt{F} \vec{x}_{1\beta_1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0.$$

   We can also see that $\vec{x}_{1\beta_1} \not\sim \vec{e}_{1\beta_1}$ and $\vec{x}_{2\beta_2} \not\sim \vec{e}_{2\beta_2}$, i.e. $x_1 \neq e_1$ and $x_2 \neq e_2$. Then, according to (1), x_1 and x_2 are projections of a single point in space.

2. $\vec{u}_{1\alpha_1} = [1,0]^\top$, $\vec{u}_{2\alpha_2} = [0,1]^\top$. We can immediately see that $x_1 = e_1$ and $x_2 = e_2$. Then, according to (2), x_1 and x_2 are projections of a single point in space.

3. $\vec{u}_{1\alpha_1} = [0,0]^\top$, $\vec{u}_{2\alpha_2} = [1,0]^\top$. We verify the epipolar constraint

$$\vec{x}_{2\beta_2}^\top \mathtt{F} \vec{x}_{1\beta_1} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1 \neq 0.$$

Then, according to the above remark, x_1 and x_2 are not projections of a single point in space.

□

Task 2. *Change one element of the matrix*

$$\mathtt{F} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

to make it a valid fundamental matrix. Find the coordinates of both epipoles in the cameras.

Solution: Since every matrix of rank 2 is a valid fundamental matrix, then it is enough to ensure the rank of $\mathtt{F}$ to be equal to 2. We may, for example, change the element $\mathtt{F}_{23}$ to 0. Then $\mathtt{F}$ becomes

$$\mathtt{F} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

The epipoles are given by the kernels of $\mathtt{F}$ and $\mathtt{F}^\top$:

$$\mathtt{F} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \ker \mathtt{F} = \left\{ \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\} = \left\langle \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{e}_{1\beta_1}} \right\rangle$$

$$\mathtt{F}^\top = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \ker \mathtt{F}^\top = \left\{ \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\} = \left\langle \underbrace{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}_{\vec{e}_{2\beta_2}} \right\rangle.$$

Remark. *Notice that it can happen that the kernel of* $\mathtt{F}$ *(or* $\mathtt{F}^\top$*) won't have a representative with the last coordinate* 1 *(all the representative will have zero there). This happens exactly when the corresponding epipole is a point at infinity. Geometrically, this means that the image plane of the corresponding camera is parallel to the baseline connecting the centers of the cameras. As a consiquence, the epipolar lines in this camera become parallel. You can encounter epipoles at infinity, e.g., in the process called "epipolar rectification", when the cameras are transformed by homographies in such a way that there image planes become parallel to the baseline. This helps after in dense reconstruction of the observed scene.*

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Task 3. *Let us have two images bound by fundamental matrix*

$$\mathtt{F} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Point X *projects in the first image into point* $[1,1]^\top$ *and in the second image on a line* $[1,1,1]^\top$*. Write the coordinates of a point, into which* X *projects in the second camera.*

Solution: We know that the epipolar line in the second camera corresponding to a point $\vec{x}_{1\beta_1}$ in the first image is given by

$$\mathbf{l} = \mathtt{F}\vec{x}_{1\beta_1} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

Then we know that the projection of X to the second camera belongs to the line given by $\mathbf{l}$. Since by the task we also know that the projection of X belongs to the line given by $\mathbf{k} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$, then the projection is given by the intersection of these lines:

$$\mathbf{x} = \mathbf{k} \times \mathbf{l} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

The fact that $\mathbf{x}$ represents a point at infinity in the second camera means geometrically that the world point X belongs to the principal plane of the second camera. □

Task 4. *Let us have two cameras with scaled camera projection matrices*

$$\mathtt{Q}_1 = \xi_1 \mathtt{P}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \mathtt{Q}_2 = \xi_2 \mathtt{P}_2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

and a point $\vec{u}_{2\alpha_2} = [1, 1]^\top$ in the second image. What are the homogeneous coordinates of the epipolar line in the first camera, that is in correspondence with the point $\vec{u}_{2\alpha_2}$?

Solution: We first normalize $\mathtt{Q}_1$ and $\mathtt{Q}_2$:

$$\mathtt{P}_1 = \frac{\operatorname{sign}|\mathtt{Q}_{1_{1:3,1:3}}|}{\left\|\mathtt{Q}_{1_{3,1:3}}\right\|} \mathtt{Q}_1 = \frac{-1}{1} \mathtt{Q}_1 = -\mathtt{Q}_1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\mathtt{P}_2 = \frac{\operatorname{sign}|\mathtt{Q}_{2_{1:3,1:3}}|}{\left\|\mathtt{Q}_{2_{3,1:3}}\right\|} \mathtt{Q}_2 = \frac{-1}{1} \mathtt{Q}_2 = -\mathtt{Q}_2 = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & -1 \end{bmatrix}$$

We decompose $\mathtt{P}_1$ and $\mathtt{P}_2$ into $\mathtt{K}_1, \mathtt{R}_1, \vec{C}_{1\delta}, \mathtt{K}_2, \mathtt{R}_2, \vec{C}_{2\delta}$ using [1, Equations (7.45)-(7.51)]:

$$\mathtt{K}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathtt{R}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \vec{C}_{1\delta} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\mathtt{K}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathtt{R}_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \vec{C}_{2\delta} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

The fundamental matrix is then given by

$$\mathtt{F} = \mathtt{K}_2^{-\top} \mathtt{R}_2 [\vec{C}_{2\delta} - \vec{C}_{1\delta}]_\times \mathtt{R}_1^\top \mathtt{K}_1^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

The epipolar line in the first camera corresponding to a point $\vec{x}_{2\beta_2}$ in the second image is given by

$$\mathbf{l} = \mathtt{F}^\top \vec{x}_{2\beta_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

□

References

[1] Tomas Pajdla, *Elements of geometry for computer vision*, https://cw.fel.cvut.cz/wiki/_media/courses/gvg/pajdla-gvg-lecture-2021.pdf.