GVG'2022 Lab-09 Solution

Task 1. *Find centers of all cameras*

$$\mathtt{P}_\beta = \begin{bmatrix} a & 0 & 1 & 0 \\ 0 & 1 & 0 & c \\ 1 & b & 1 & 0 \end{bmatrix}$$

which project point $[1,1,1]^\top$ *in space into point* $[1,1]^\top$ *in the image.*

Solution: First of all, for $\mathtt{P}_\beta$ to be a valid image projection matrix it must take the form

$$\mathtt{P}_\beta = \left[\mathtt{A} \mid -\mathtt{A}\vec{C}_\delta\right]$$

where $\mathtt{A}$ is invertible 3×3 matrix. Thus, there is a restriction on $\mathtt{P}_\beta$:

$$\det \mathtt{P}_{\beta_{1:3,1:3}} \neq 0 \iff a \neq 1.$$

By definition, a world point X projects into a point $[u,v]^\top$ in the image if there exists a unique line connecting X and the camera projection center C and this line intersects the image plane in $\mathbb{A}^3$ at x with $x_{(o,\alpha)} = [u,v]^\top$. This geometric definition may be rewritten algebraically in the equivalent form as follows: a world point X projects into a point $[u,v]^\top$ in the image if

$$\exists\, \eta \in \mathbb{R}\backslash\{0\} : \quad \eta \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathtt{P}_\beta \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \tag{1}$$

Remark. *Notice that the statement*

$$\exists \eta \in \mathbb{R} : \quad \eta \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathtt{P}_\beta \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \tag{2}$$

is not equivalent to (1). *It is true that* (1) $\Rightarrow$ (2) *since if* $\eta \in \mathbb{R}\backslash\{0\}$, *then* $\eta \in \mathbb{R}$. *However, the converse* (2) $\Rightarrow$ (1) *doesn't hold. To see why, take* $X = C$. *Then the right hand side of both* (1) *and* (2) *becomes the zero vector. While in* (2) *we can take* $\eta = 0$ *to make the matrix equation true, in* (1) *there is no such* η. *(In other words,* (2) *also enables* C *"to be projected" to the image point* $[u,v]^\top$, *while* (1) *does not.)*

Substituting known values to (1) we obtain

$$\eta \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a & 0 & 1 & 0 \\ 0 & 1 & 0 & c \\ 1 & b & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \eta \in \mathbb{R}\backslash\{0\}$$

$$\begin{cases} \eta = a + 1 \\ \eta = c + 1 \\ \eta = b + 2 \end{cases}, \quad \eta \in \mathbb{R}\backslash\{0\}$$

$$\begin{cases} a = \eta - 1 \\ b = \eta - 2 \\ c = \eta - 1 \end{cases}, \quad \eta \in \mathbb{R}\backslash\{0\}$$

Substituting a, b, c into $\mathtt{P}_\beta$ (and remembering that $a \neq 1$) we get the set S of all possible cameras

$$S = \left\{ \begin{bmatrix} \eta - 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \eta - 1 \\ 1 & \eta - 2 & 1 & 0 \end{bmatrix} \;\middle|\; \eta \in \mathbb{R}\backslash\{0, 2\} \right\}$$

which project point $[1,1,1]^\top$ in space into point $[1,1]^\top$ in the image. To find centers of these cameras we need to invert the left 3×3 block parametrized by η:

$$\begin{bmatrix} \eta-1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & \eta-2 & 1 \end{bmatrix}^{-1} = \frac{1}{\eta-2}\begin{bmatrix} 1 & 0 & -1 \\ \eta-2 & \eta-2 & -\eta^2+3\eta-2 \\ -1 & 0 & \eta-1 \end{bmatrix}^\top = \frac{1}{\eta-2}\begin{bmatrix} 1 & \eta-2 & -1 \\ 0 & \eta-2 & 0 \\ -1 & -\eta^2+3\eta-2 & \eta-1 \end{bmatrix}$$

$$\vec{C}_\delta = -\mathtt{P}_{\beta_{1:3,1:3}}^{-1}\mathtt{P}_{\beta_{1:3,4}} = -\frac{1}{\eta-2}\begin{bmatrix} 1 & \eta-2 & -1 \\ 0 & \eta-2 & 0 \\ -1 & -\eta^2+3\eta-2 & \eta-1 \end{bmatrix}\begin{bmatrix} 0 \\ \eta-1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-\eta \\ 1-\eta \\ (1-\eta)^2 \end{bmatrix}$$

Thus, the set of camera centers of all cameras from S is described by

$$\left\{ \begin{bmatrix} 1-\eta \\ 1-\eta \\ (1-\eta)^2 \end{bmatrix} \;\middle|\; \eta \in \mathbb{R}\backslash\{0,2\} \right\}.$$

□

Task 2. *Let us have two vanishing points in the image represented by vectors* $\vec{u}_{1\alpha} = [0,0]^\top$ *and* $\vec{u}_{2\alpha} = [2,0]^\top$, *which come from the image of an observed rectangle. Find all values of parameter* a *in the matrix*

$$\mathtt{K} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

of a camera which captured the image.

Solution: Let us denote by $\vec{x}_{1\beta} = [0,0,1]^\top$ and $\vec{x}_{2\beta} = [2,0,1]^\top$ the two vectors representing given vanishing points in the camera coordinate system (C,β). Since the given vanishing points are images of points at infinity of two perpendicular lines in the world, then $\vec{x}_1 \perp \vec{x}_2$. To express this constraint algebraically we need to pass to the coordinates of $\vec{x}_1$ and $\vec{x}_2$ in some orthogonal basis (e.g. γ):

$$\vec{x}_{1\gamma}^\top \vec{x}_{2\gamma} = 0 \iff \vec{x}_{1\beta}^\top \mathtt{K}^{-\top}\mathtt{K}^{-1}\vec{x}_{2\beta} = 0$$

We compute

$$\mathtt{K}^{-1} = \begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathtt{K}^{-\top} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & 0 & 1 \end{bmatrix}, \quad \mathtt{K}^{-\top}\mathtt{K}^{-1} = \begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & 0 \\ -a & 0 & a^2+1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & 0 \\ -a & 0 & a^2+1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 0$$

$$a^2 - 2a + 1 = 0 \iff a = 1.$$

□

Task 3. *Consider the homography with the following matrix*

$$\mathtt{H} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & a & 1 \end{bmatrix}$$

Find the parameter a, *to get point in the image represented by* $\vec{u}_\alpha = [1,1]^\top$ *mapped into a point at infinity.*

Solution: The condition in the task may be rewritten algebraically as follows:

$$\lambda \begin{bmatrix} u \\ v \\ 0 \end{bmatrix} = \mathtt{H} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \lambda \neq 0,\ u \neq 0 \text{ or } v \neq 0$$

We can reparametrize the variables using substitution $u' = \lambda u$, $v' = \lambda v$. Then conditions $\lambda \neq 0,\ u \neq 0$ or $v \neq 0$ will be equivalently rewritten as $u' \neq 0$ or $v' \neq 0$. Thus, we have

$$\begin{bmatrix} u' \\ v' \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & a & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad u' \neq 0 \text{ or } v' \neq 0$$

$$\begin{cases} u' = 2 \\ v' = 1 \\ a + 1 = 0 \end{cases}, \quad u' \neq 0 \text{ or } v' \neq 0$$

Hence $a = -1$. We can see that

$$\mathtt{H} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

is a valid homography matrix (i.e. it is invertible). □

Task 4. *Consider line l in $\mathbb{P}^2$ represented by $\mathbf{l} = [1, 0, 1]^\top$ and homography*

$$\mathtt{H} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which maps line l onto line l'. Find the point on the line l that is mapped onto itself by $\mathtt{H}$.

Solution: We first determine all the points in $\mathbb{P}^2$ that are mapped onto themselves by $\mathtt{H}$:

$$\lambda \mathbf{x} = \mathtt{H}\mathbf{x}, \quad [\mathbf{x}] \in \mathbb{P}^2,\ \lambda \neq 0$$

This may be equivalently restated as finding eigenvectors of $\mathtt{H}$ (since $\mathtt{H}$ is invertible, then all its eigenvalues are nonzero). We first find the eigenvalues of $\mathtt{H}$:

$$\det(\lambda \mathtt{I} - \mathtt{H}) = 0 \iff (\lambda - 1)^3 = 0 \iff \lambda = 1.$$

To find the eigenspace corresponding to the eigenvalue $\lambda = 1$ we solve

$$(1 \cdot \mathtt{I} - \mathtt{H})\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

The set of solutions is a 2-dimensional linear space:

$$S = \left\langle \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{x}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{x}_2} \right\rangle$$

In other words, every point in $\mathbb{P}^2$ in the form $a\mathbf{x}_1 + b\mathbf{x}_2$ for $a, b \in \mathbb{R}$ is mapped onto itself by $\mathtt{H}$. Notice that all these points are points at infinity, since the last coordinates of $\mathbf{x}_1$ and $\mathbf{x}_2$ (and thus of $a\mathbf{x}_1 + b\mathbf{x}_2$ for $a, b \in \mathbb{R}$) are zero. These points form the line at infinity k in $\mathbb{P}^2$ represented by

$$\mathbf{k} = \mathbf{x}_1 \times \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

In order to find a point on the line l that is mapped onto itself by $\mathtt{H}$ we need to find the intersection of k and l:

$$\mathbf{p} = \mathbf{k} \times \mathbf{l} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

□

Task 5. *Find all points in $\mathbb{P}^2$, which are projected into themselves by homography*

$$\mathtt{H} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: See the first part of the solution to Task 4. □

Task 6. *Consider points $\mathbf{x} = [1, 0, 1]^\top$, $\mathbf{y} = [1, 2, 0]^\top$ and $\mathbf{z} = [0, 1, 1]^\top$ in the real projective plane. Find the line l which is parallel (in the canonically associated affine plane) to the line passing through points $\mathbf{x}, \mathbf{y}$ and such that l passes through $\mathbf{z}$.*

Solution: The fact that l is parallel (in the canonically associated affine plane) to the line l' passing through points $\mathbf{x}, \mathbf{y}$ means that l and l' meet at a point at infinity. Since $y \in l'$ and the last coordinate of the representative $\mathbf{y}$ of y is zero, then $l \cap l' = y$, or $y \in l$. Since $z \in l$ by the task, then l is a line passing through y and z:

$$\mathbf{l} = \mathbf{y} \times \mathbf{z} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$$

□

Task 7. *Find all constraints on parameters a, b such that the homography represented by*

$$\mathtt{H} = \begin{bmatrix} a & 0 & 1 \\ b & 0 & 1 \\ a & b & 1 \end{bmatrix}$$

maps line $\mathbf{l} = [0, 1, 1]^\top$ onto the line at infinity.

Solution: First of all, for $\mathtt{H}$ to be a valid homography matrix it must be invertible, i.e.

$$\det \mathtt{H} \neq 0 \iff b(b - a) \neq 0 \iff b \neq 0 \text{ and } a \neq b.$$

Suppose that $x, y \in l$, whose homogeneous coordinates in $\mathbb{P}^2$ are $\mathbf{x}$ and $\mathbf{y}$. Using the property of the cross product we can write

$$\underbrace{\mathtt{H}\mathbf{x} \times \mathtt{H}\mathbf{y}}_{\mathbf{l}'} = \frac{1}{\det \mathtt{H}^{-\top}} \mathtt{H}^{-\top} \underbrace{(\mathbf{x} \times \mathbf{y})}_{\mathbf{l}}$$

This means that having a line l in $\mathbb{P}^2$ with homogeneous coordinates $\mathbf{l}$ and a homography matrix $\mathtt{H}$, the homogeneous coordinates $\mathbf{l}'$ of the image l' of l by $\mathtt{H}$ may be obtained by $\mathtt{H}^{-\top}\mathbf{l}$ (since homogeneous coordinates are defined up to scale, we may forget about the scale $\frac{1}{\det \mathtt{H}^{-\top}}$).

The condition in the task may be rewritten algebraically as follows:

$$\lambda \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathtt{H}^{-\top} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \lambda \neq 0$$

We compute

$$\mathtt{H}^{-\top} = \frac{1}{b(b - a)} \begin{bmatrix} -b & a - b & b^2 \\ b & 0 & -ab \\ 0 & b - a & 0 \end{bmatrix}$$

Hence

$$\lambda \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{b(b-a)} \begin{bmatrix} -b & a-b & b^2 \\ b & 0 & -ab \\ 0 & b-a & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \lambda \neq 0$$

$$\begin{cases} 0 = a - b + b^2 \\ 0 = -ab \\ b(b-a)\lambda = b - a \end{cases}, \quad \lambda \neq 0$$

From the second equation $0 = -ab$ we conclude that $a = 0$, since $b \neq 0$. Substituting $a = 0$ into the first equation we get $0 = -b + b^2$ which means that $b = 1$ (since $b \neq 0$). We still need to verify if there is a nonzero solution to λ. For this we substitute $a = 0$ and $b = 1$ to the last equation and get $\lambda = 1$. Thus, $a = 0$ and $b = 1$ is indeed a solution which generates a valid homography matrix

$$\mathtt{H} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

□