Reasons of Introducing the Language of Projective Geometry

GVG Lab 08

April 2022

Projection from $\mathbb{A}^3$ to $\mathbb{A}^2$

The algebraic model of perspective projection from (almost whole) $\mathbb{A}^3$ to $\mathbb{A}^2$ has the form

$$\eta \begin{bmatrix} \vec{u}_\alpha \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \mathtt{A} \mid -\mathtt{A}\vec{C}_\delta \end{bmatrix}}_{\mathtt{P}_\beta} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}, \quad \eta \neq 0$$

It assumes that X doesn't belong to the principal plane.

What about points from the principal plane?

We can still evaluate the product $\mathtt{P}_\beta \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}$ on the right-hand side for X from the principal plane to see what happens:

$$\begin{bmatrix} u \\ v \\ 0 \end{bmatrix} = \mathtt{P}_\beta \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}$$

We can see that the vector on the left doesn't have a representation as $\eta \begin{bmatrix} \vec{u}_\alpha \\ 1 \end{bmatrix}$ for $\eta \neq 0$.

Extending affine plane $\mathbb{A}^2$ to projective plane $\mathbb{P}^2$

Geometric construction: Identify points in the image plane with rays passing through those points and the camera center (**finite points** of the projective plane, or points that are **visible in the image**), and add new rays passing through the camera center and lying in the principal plane (**ideal points** or **points at infinity** of the projective plane, or points that are **not visible in the image**).

Algebraic construction: For the equivalence relation $\sim$ on the set $\mathbb{R}^3 \backslash \{\mathbf{0}\}$ defined by

$$\mathbf{x}_1 \sim \mathbf{x}_2 \iff \exists \lambda \in \mathbb{R} \backslash \{0\} : \mathbf{x}_1 = \lambda \mathbf{x}_2$$

we define

$$\mathbb{P}^2 = \left(\mathbb{R}^3 \backslash \{\mathbf{0}\} \right) / \sim \; = \left\{ [\mathbf{x}] \mid \mathbf{x} \in \mathbb{R}^3 \backslash \{\mathbf{0}\} \right\},$$

where

$$[\mathbf{x}] = \{ \lambda \mathbf{x} \mid \lambda \in \mathbb{R} \backslash \{0\} \}$$

is a **point** of the projective plane $\mathbb{P}^2$.

Extending affine plane $\mathbb{A}^2$ to projective plane $\mathbb{P}^2$

Advantages:

1) Now any two lines in $\mathbb{P}^2$ intersect (even parallel ones) $\rightarrow$ studying points and lines becomes simpler;

2) Working with just 1 point (the camera center) which doesn't project to the camera is easier than with the plane of points which don't project (the principal plane). As a consequence, the back-projected plane of an image line is a plane in $\mathbb{A}^3$ with just 1 point removed (the camera center);

3) Despite the fact that points at infinity of $\mathbb{P}^2$ are not visible in the image, we can still get useful information from them algebraically, e.g. using vanishing points at infinity for camera calibration.

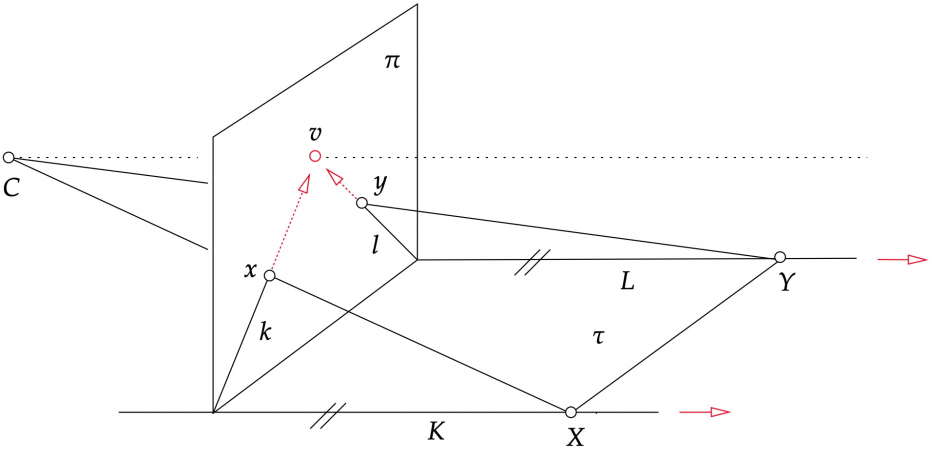
Projection from $\mathbb{A}^3$ to $\mathbb{P}^2$

By extending $\mathbb{A}^2$ to $\mathbb{P}^2$ we extend the domain of definition of the projection map:

$$\mathbf{x} = \mathtt{P}_\beta \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}, \quad [\mathbf{x}] \in \mathbb{P}^2$$

Now X can be any point from $\mathbb{A}^3$ except for the camera projection center C.

Vanishing point



Extending affine space $\mathbb{A}^3$ to projective space $\mathbb{P}^3$

Reason: It happens that vanishing points can be used for camera calibration. We can introduce 3 different definitions for the vanishing point:

(a) the intersection of the projections of 2 parallel lines;

(b) the limit of the projection of a line as the variable which parametrizes it goes to infinity;

(c) **the projection of the point at infinity of a line**.

(Of course, the last definition only makes sense after introducing the projective space). From some point of view the last definition is the most convenient to work with.

Idea: identify points of $\mathbb{A}^3$ with 1D subspaces of $\mathbb{A}^4$ generated by $\begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}$ (**finite points** of the projective space) and add 1D subspaces generated by $\begin{bmatrix} \mathbf{d} \\ 0 \end{bmatrix}$ (**points at infinity** of the projective space).

Projection from $\mathbb{P}^3$ to $\mathbb{P}^2$

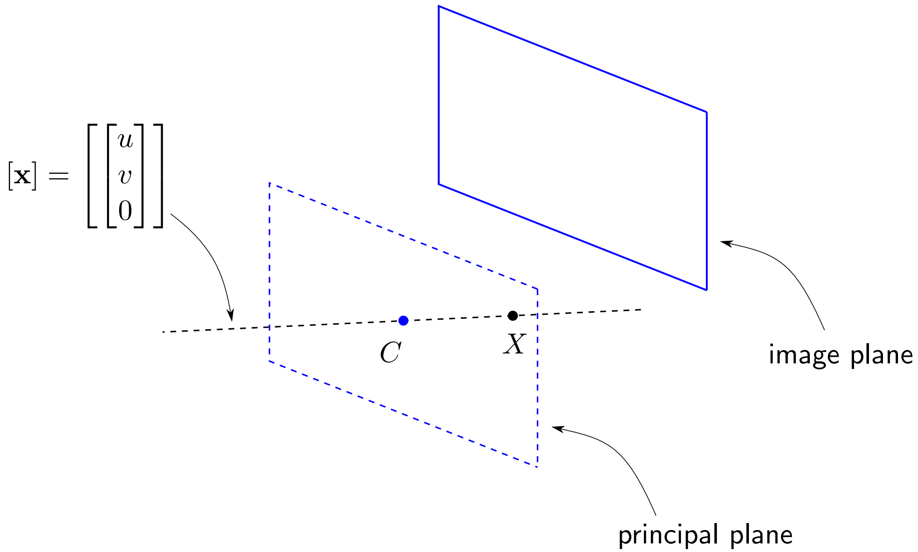
By extending $\mathbb{A}^3$ to $\mathbb{P}^3$ we extend the domain of the projection map:

$$\mathbf{x} = \mathtt{P}_\beta \mathbf{X}, \quad [\mathbf{x}] \in \mathbb{P}^2, [\mathbf{X}] \in \mathbb{P}^3$$

The world point X can now be any point from $\mathbb{P}^3$ (i.e. including points at infinity) except for the camera projection center C.

Finite points of $\mathbb{P}^3 \to$ finite points of $\mathbb{P}^2$

Finite points of $\mathbb{P}^3 \rightarrow$ points at infinity of $\mathbb{P}^2$



Points at infinity of $\mathbb{P}^3 \to$ finite points of $\mathbb{P}^2$

Points at infinity of $\mathbb{P}^3 \to$ points at infinity of $\mathbb{P}^2$

Intersection of 2 lines in $\mathbb{P}^2$

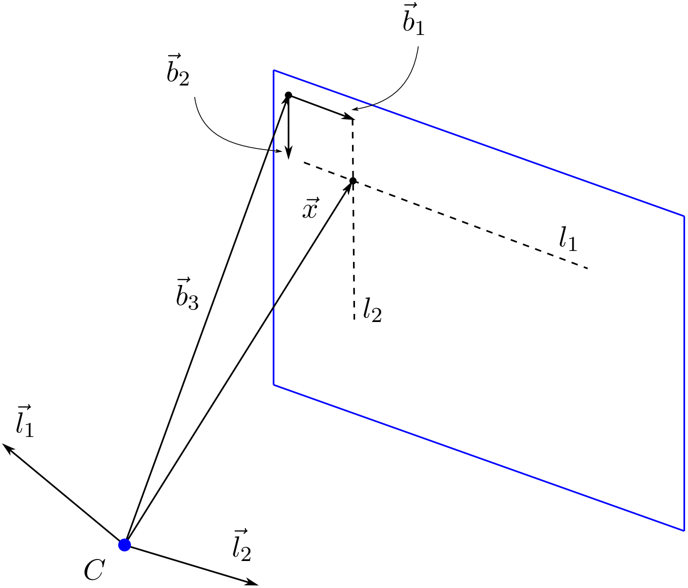
Task

Let us have two lines in the image l_1 and l_2 given by:

$$l_1: \; v = 1, \quad l_2: \; u = 1.$$

Find their intersection (using techniques of projective geometry).

Intersection of 2 lines in $\mathbb{P}^2$



Line passing through 2 points in $\mathbb{P}^2$

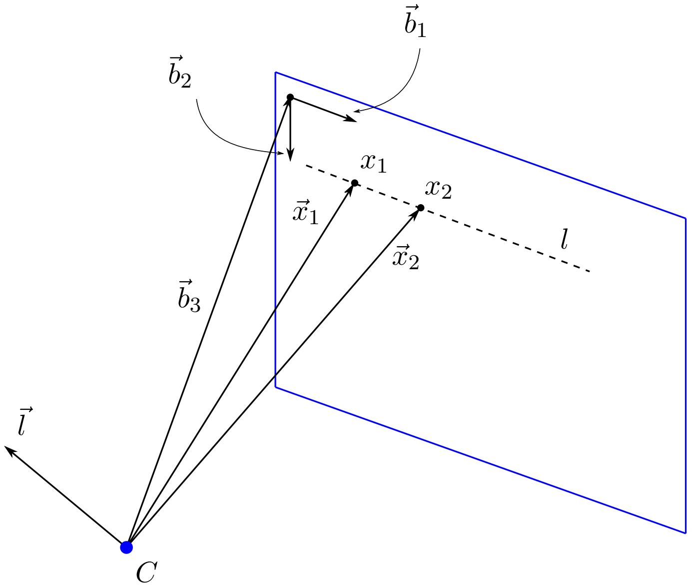
Task

Let us have two image points x_1 and x_2 defined by

$$\vec{u}_{1\alpha} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{u}_{2\alpha} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Find the line in the image passing through them (using techniques of projective geometry).

Line passing through 2 points in $\mathbb{P}^2$



Lines in $\mathbb{P}^3$

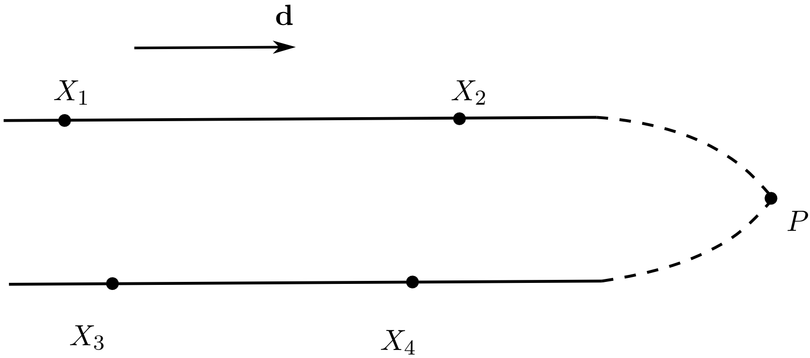
Task

Let us have two lines L_1 and L_2 in $\mathbb{A}^3$ given by:

$$L_1: \ \vec{X}_{1\delta} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{X}_{2\delta} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}; \quad L_2: \ \vec{X}_{3\delta} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{X}_{4\delta} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Find the intersection of L_1 and L_2 (if exists) in the projective space $\mathbb{P}^3$.

Lines in $\mathbb{P}^3$



Vanishing points and horizon

Task

Let the camera be given by the following camera projection matrix

$$\mathtt{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Let the rectangle in space be defined by the following 4 points:

$$\vec{X}_{1\delta} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{X}_{2\delta} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \vec{X}_{3\delta} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \vec{X}_{4\delta} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Find the horizon of the plane defined by the rectangle.

Vanishing points and horizon