Probabilistic classification

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Notes

(Re-)introduction uncertainty/probability

- Markov Decision Processes (MDP) – uncertainty about outcome of actions
- Now: uncertainty may be also associated with states
  - Different states may have different prior probabilities.
  - The states $s \in \mathcal{S}$ may not be directly observable.
  - They need to be inferred from features $x \in \mathcal{X}$.
- This is addressed by the rules of probability *(such as Bayes theorem)* and leads on to
  - Bayesian classification
  - Bayesian decision making

Notes

Just a reminder: MDPs, value iteration and policy iteration methods. We were looking for an optimal policy $\pi : \mathcal{S} \to \mathcal{A}$.

red and blue boxes; apples and oranges

Dark warehouse, color not directly observable. Getting stats:

- Pick box at random, pull it out to the light.
- Pick a fruit from the box, at random.

Random variables: B box color, F fruit kind

Notes

Joint probabilities $P(B, F)$: $P(r \wedge a) = P(r, a) = \frac{2}{20}$ similarly for 3 remaining combinations.

Marginal probabilites $P(B), P(F)$:
$P(a) = \sum_{B=r,b} P(a, B) = \frac{2}{20} + \frac{9}{20} = \frac{11}{20}$
Similarly for all, always summing along the other one.

Conditional probabilities $P(F|B) = \frac{P(F,B)}{P(B)}$: Fixing one row or column, hiding the rest. Similarly, $P(B|F) = \frac{P(F,B)P(F)}{P(F)}$. Obviously:

$$P(F, B) = P(B|F)P(F) = P(F|B)P(B)$$

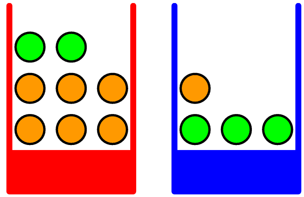
Probability of B or F:
$P(B \vee F) = P(B) + P(F) - P(B \wedge F)$

Fruit

Box	a	o
r	2/20 ||	|||||| 6/20
b	9/20 |||||||||	||| 3/20

Probability example: Picking fruits

- red box: 2 apples, 6 oranges
- blue box: 3 apples, 1 orange



- Scenario: Pick a box—say red box in 40% cases. *Then* pick a fruit at random.
- (Frequent) questions:
  - What is the overall probability that the selection procedure will pick an apple?
  - Given that we have chosen an orange, what is the probability that it was from the blue box?

Example from Chapter 1.2 [1]

Notes

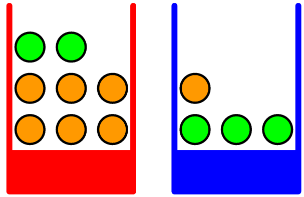
Fruit

Box

	a	o
r	2/20 ||	|||||| 6/20
b	9/20 |||||||||	||| 3/20

Picking fruits. What is the probability that ...?

- red box: 2 apples, 6 oranges
- blue box: 3 apples, 1 orange



Procedure: Pick a box (say red box in 40% cases), then pick a fruit at random.

Quiz 1: What is the probability that the selection procedure will pick an apple?

A: 11/20

B: 6/8

C: 1/2

D: Different value.

Notes

Example serves for probability recap (sum, product rules, conditional probabilities, Bayes)
Random variables:

- Identity of the box B, two possible values r, b
- Identity of the fruit F, two possible values a, o

Info about picking a box:

- $P(B = r) = 0.4$
- $P(B = b) = 0.6$

Conditional probabilities, given box selected: $P(o|r) = 3/4, P(a|r) = 1/4, P(o|b) = 1/4, P(a|b) = 3/4$. Answering questions:

- $P(F = a) = P(a|r)P(r) + P(a|b)P(b) = (2/8) * (4/10) + (3/4) * (6/10) = 11/20$
- $P(B = b|F = o) = P(b|o)$

$$P(b|o) = \frac{P(o|b)P(b)}{P(o)} = \frac{P(o|b)P(b)}{P(o|b)P(b) + P(o|r)P(r)} = 1/3$$

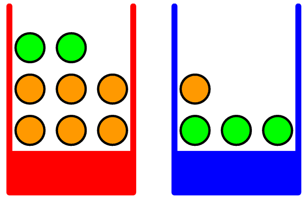
or $P(o) = 1 - P(a) = 1 - 11/20 = 9/20$

$P(B)$ prior probability – *before* we observe the fruit.

$P(B|F)$ aposteriori probability – *after* we observe the fruit.

Picking fruits. What is the probability that ...?

- red box: 2 apples, 6 oranges
- blue box: 3 apples, 1 orange



Procedure: Pick a box (say red box in 40% cases), then pick a fruit at random.

Quiz 2: Given that we have chosen an orange, what is the probability that it was from the blue box?

A: 1/4

B: 3/5

C: 1/3

D: Different value.

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Rules of probability and notation I

- random variables X, Y
- x_i where $i = 1, ..., M$ – values taken by variable X
- y_j where $j = 1, ..., L$ – values taken by variable Y
- $P(X = x_i, Y = y_i)$ – probability that X takes the value x_i *and* Y takes y_i – joint probability
- $P(X = x_i)$ – probability that X takes the value x_i
- Sum rule of probability :
  - $P(X = x_i) = \sum_{j=1}^{L} P(X = x_i, Y = y_j)$
  - $P(X = x_i)$ is sometimes called marginal probability – obtained by marginalizing / summing out the other variables
  - general rule, compact notation: $P(X) = \sum_Y P(X, Y)$

Notes

This and the following slides are just to formally recap what we learned when discussing boxes and fruits.

Rules of probability and notation II

- **Conditional probability** : $P(Y = y_j | X = x_i)$
- **Product rule of probability** :
  - $P(X = x_i, Y = y_i) = P(Y = y_j | X = x_i) P(X = x_i)$
  - general rule, compact notation: $P(X, Y) = P(Y|X)P(X)$
- **Bayes theorem** :
  - from $P(X, Y) = P(Y, X)$ and product rule

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

$$\begin{aligned} P(disease|symptoms) &= \frac{P(symptoms|disease) \times P(disease)}{P(symptoms)} \\ posterior &= \frac{likelihood \times prior}{evidence} \end{aligned}$$

- **Independence** : $P(X, Y) = P(X)P(Y)$

Notes

What does is mean when we say that random variables X and Y are independent?

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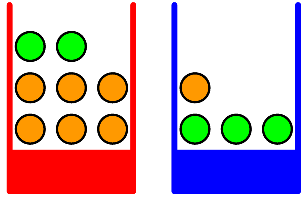
Boxes and **F**ruits: posterior? likelihood? prior? evidence?

$$posterior = \frac{likelihood \times prior}{evidence}$$



Connect with lines:

- posterior
  after observation
- likelihood
  of an observation
- prior
  before observation
- evidence
  total observations

- $P(B)$
- $P(F)$
- $P(F \mid B)$
- $P(B \mid F)$

Notes

Boxes and **F**ruits:

- prior (before observation) – $P(B)$
- likelihood (of observation) – $P(F|B)$
- evidence (total observations) – $P(F)$
- posterior (after observation) – $P(B|F)$

Think about these terms—it helps to understand and remember.

Decision example: Insure or not? (from late 1980s) [4]

A doctor calls: “Your HIV test is positive, 999/1000 you will die in 10 years. I’m sorry …”. Insurance company does not want to insure a married couple.

- Was the doctor right?
- Was the insurance company rational?

What the doctor (and the company) knew:

- HIV test falsely positive only in 1 case out of 1000.

Notes

Equations/formulas are simple but not easy to (fully) understand.

- Doctor: $P(\text{positive test} \mid \text{healthy}) = \frac{1}{1000}$ but this is the *likelihood* which we learn before the patient’s diagnosis (classification).
- More interesting and important is to know: $P(\text{healthy} \mid \text{positive test})$ (*posterior*).
- Think about 10000 samples of heterosexual males, family, …. Statistically, there is just 1 HIV positive among them.
- Assume $P(\text{negative test} \mid \text{infected}) \to 0$. (false negative rate)
- 1 person HIV positive will be tested positive, but also 10 other healthy persons will be tested positive. Hence $P(\text{healthy} \mid \text{positive test}) = 10/11$.
- Or, for the doctor: $P(\text{infected} \mid \text{positive test}) = \frac{1}{11}$ and not $\frac{999}{1000}$.
- The fact that a disease is rare matters a lot!

Symptoms (test)

State		Positive	Negative
	healthy	1/1000	
	ill infected	~~999/1000~~ ? no!	

Think about 10000 samples (individuals). Among white male heterosexuals, HIV is a rare disease.

Symptoms (test)

State		Positive	Negative
	healthy	10/(10000)	9989
	ill infected	1	∅

$P(\text{ill} \mid \text{pos}) = \frac{1}{11}$!

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Insurance company does not want to insure a married couple.

- Was the doctor right?
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What the doctor (and the company) knew:

- HIV test falsely positive only in 1 case out of 1000.

What is the probability the man is infected?

A: $\frac{1}{1000}$

B: $\frac{999}{1000}$

C: Don't know yet, more info needed, but less than $\frac{1}{2}$

D: Don't know yet, more info needed, but more than $\frac{1}{2}$

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What the doctor (and the company) knew:

- HIV test falsely positive only in 1 case out of 1000.
- Heterosexual male, has family, no drugs, no risk behavior.

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Decision: guilty or not? (people of CA vs Collins, 1968) [4]

- Robbery, LA 1964, fuzzy evidence of the offenders:
  - female, around 65 kg
  - wearing something dark
  - hair of light color, between light and dark blond, in a ponytail
- At the same time, additional evidence close to the crime scene:
  - loud scream, yelling, looking at the this direction ...
  - a woman sitting into a yellow car
  - car starts immediately and passes close to the additional witness
  - a black man with beard and moustache was driving
- No more evidence
- Testimony of both the victim and the witness not unambiguous (didn't recognize suspects)
- Still, the suspects were sentenced to jail.

Notes

Wrong use of independence assumption:

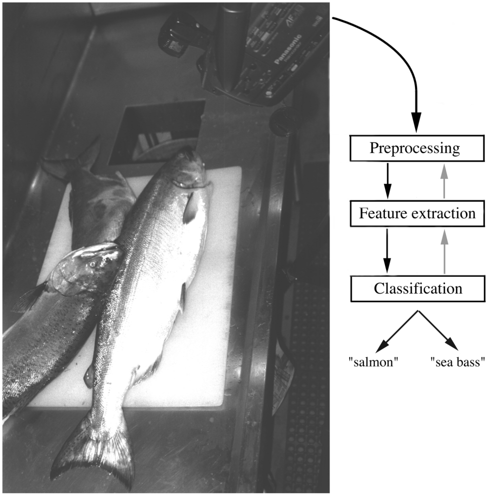
$$\begin{aligned} P(\text{yellow car}) &= 1/10 \\ P(\text{man with moustache}) &= 1/4 \\ P(\text{black man with beard}) &= 1/10 \\ P(\text{woman with pony tail}) &= 1/10 \\ P(\text{woman blond hair}) &= 1/3 \\ P(\text{mix race pair in a car}) &= 1/1000 \end{aligned}$$

and mistakenly confusing probability

$$P(\text{randomly selected pair matches discussed characteristics})$$

giving $P = 1/12000000$. Think about total Cal poulation.
with the needed conditional probability: $P(\text{a pair matching characteristics is guilty})$
"The court noted that the correct statistical inference would be the probability that no other couple who could have committed the robbery had the same traits as the defendants given that at least one couple had the identified traits. The court noted, in an appendix to its decision, that using this correct statistical inference, even if the prosecutor's statistics were all correct and independent as he assumed, the probability that the defendants were innocent would be over 40%." https://en.wikipedia.org/wiki/People_v._Collins

Classification example: What's the fish?



- Factory for fish processing
- 2 classes $s_{1,2}$:
  - salmon
  - sea bass
- Features $\vec{x}$: length, width, lightness etc. from a camera

Notes

- Sea (European) bass, https://en.wikipedia.org/wiki/European_bass. (In Czech it is Mořčák evropský or Mořský vlk.)
- Salmon, https://en.wikipedia.org/wiki/Salmon. (losos in Czech)

Fish – classification using probability

$$posterior = \frac{likelihood \times prior}{evidence}$$

- Notation for classification problem
  - Classes $s_j \in \mathcal{S}$ (e.g., salmon, sea bass)
  - Features $x_i \in \mathcal{X}$ or feature vectors ($\vec{x_i}$) (also called attributes)
- Optimal classification of $\vec{x}$:

$$\delta^*(\vec{x}) = \arg\max_j P(s_j|\vec{x})$$

- We thus choose the most probable class for a given feature vector .
- Both likelihood and prior are taken into account – recall Bayes rule:

$$P(s_j|\vec{x}) = \frac{P(\vec{x}|s_j)P(s_j)}{P(\vec{x})}$$

- Can we do (classify) better?

Notes

Assuming we know the true $P(\vec{x}|s_j), P(s_j), P(\vec{x})$ we *cannot* do better! Bayesian classification is optimal!

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Decision making under uncertainty

- An important feature of intelligent systems
  - make the best possible decision
  - in uncertain conditions
- **Example:** Take a tram OR subway from A to B?
  - Tram: timetables imply a quicker route, but adherence uncertain.
  - Subway: longer route, but adherence almost certain.
- **Example:** where to route a letter with this ZIP?
  - 15700? 15706? 15200? 15206?
- What is the optimal decision ?
- What is the cost of the decision? What is the associated loss ?
- What is the relation between loss and utility ?

Notes

There are *costs* associated with a decision. E.g. at fish packing plant, customers may not mind so much if some pieces of salmon end up in sea bass cans, but they will be protesting if the opposite happens. So making an error "one way" has higher cost than "the other way". This impacts where decision boundaries for classification should optimally be drawn.

The decision loss can be seen as counterpart of the utility . We want either maximize utitity or minimize loss. In machine learing and pattern recoginition community, the term loss is used much more frequently.

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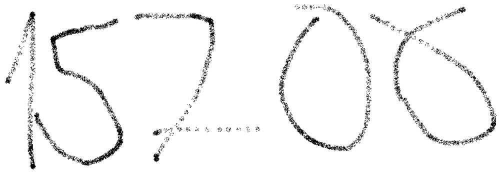

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There are *costs* associated with a decision. E.g. at fish packing plant, customers may not mind so much if some pieces of salmon end up in sea bass cans, but they will be protesting if the opposite happens. So making an error “one way” has higher cost than “the other way”. This impacts where decision boundaries for classification should optimally be drawn.

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Decision making under uncertainty

- An important feature of intelligent systems
  - make the best possible decision
  - in uncertain conditions
- **Example**: Take a tram OR subway from A to B?
  - Tram: timetables imply a quicker route, but adherence uncertain.
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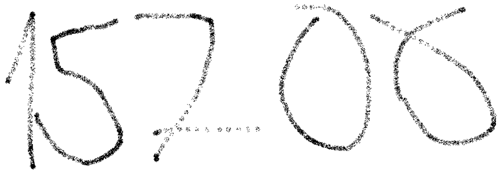
- **Example**: where to route a letter with this ZIP?



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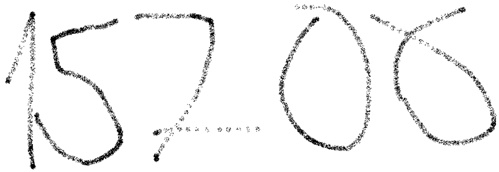

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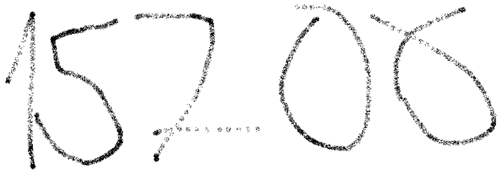

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Introducing decision loss: What to cook for dinner [3]

- *Wife is coming back from work. Husband: what to cook for dinner?*
- 3 dishes (decisions) in his repertoire:
  - *nothing* … **don't bother cooking** ⇒ no work but makes wife upset
  - *pizza* … **microwave a frozen pizza** ⇒ not much work but won't impress
  - *g.T.c.* … **general Tso's chicken** ⇒ will make her day, but very laborious
- "Hassle" incurred by the individual options depends on wife's mood.
- For each of the 9 possible situations (3 possible decisions × 3 possible states), the cost is quantified by a loss function $l(d, s)$:

$l(s, d)$	$d = nothing$	$d = pizza$	$d = g.T.c.$
$s = good$	0	2	4
$s = average$	5	3	5
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The wife's state of mind is an uncertain state.

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Was the state known, the decision would be simple.

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Example (cont'd), State uncertain, symptoms, ...

- Husband's experiment. He tells her he accidentally overtaped their wedding video and observes her reaction.
- Anticipates 4 possible reactions:
  - *mild* ... all right, we keep our memories.
  - *irritated* ... how many times do I have to tell you....
  - *upset* ... Why did I marry this guy?
  - *alarming* ... silence
- The reaction is a measurable attribute/symptom ("feature") of the mind state.
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Decision strategy

- **Decision strategy** : a rule selecting a decision for *any given value* of the measured attribute(s).
- i.e. function $d = \delta(x)$.
- Example of husband's possible strategies:

$\delta(x)$	$x = mild$	$x = irritated$	$x = upset$	$x = alarming$
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$\delta_4(x) =$	*nothing*	*nothing*	*nothing*	*nothing*

- How many strategies?
- How to define which strategy is the best? How to sort them by quality?
- Define the risk of a strategy as a mean (expected) loss value.

$$r(\delta) = \sum_x \sum_s l(s, \delta(x)) P(x, s)$$

Notes

Overall, $3^4 = 81$ possible strategies (3 possible decisions for each of the 4 possible attribute values). There is some analogy of states and possible actions. Here, we reason about states - which are 3 (state of mind) - from features which are 4.
Any given value (of measured attribute) ... Think about any possible state.
Recall MDPs and RL.

- Reward (or penalty) was associated with state or state transition when executing an action $R(s, a, s')$. Similarly here, loss, $l(s, \delta(x))$, is associated with state and decision/action.
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Bayes optimal strategy

- The Bayes optimal strategy : one minimizing mean risk.

$$\delta^* = \arg\min_{\delta} r(\delta)$$

- From $P(x,s) = P(s|x)P(x)$ (Bayes rule), we have

$$r(\delta) = \sum_x \sum_s l(s,\delta(x))P(x,s) = \sum_s \sum_x l(s,\delta(x))P(s|x)P(x)$$
$$= \sum_x P(x) \underbrace{\sum_s l(s,\delta(x))P(s|x)}_{\text{Conditional risk}}$$

- The optimal strategy is obtained by minimizing the conditional risk *separately* for each x:

$$\delta^*(x) = \arg\min_{d} \sum_s l(s,d)P(s|x)$$

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Optimal strategy: $\delta^*(x) = \arg\min_d \sum_s l(s,d)P(s|x)$

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$P(x,s)$	$x =$ *mild*	$x =$*irritated*	$x =$ *upset*	$x =$ *alarming*
$s =$ *good*	0.35	0.28	0.07	0.00
$s =$ *average*	0.04	0.10	0.04	0.02
$s =$ *bad*	0.00	0.02	0.05	0.03

$\delta(x)$	$x =$ *mild*	$x =$ *irritated*	$x =$ *upset*	$x =$ *alarming*
$\delta^*(x) =$	??	??	??	??

Notes

We need to recompute the table of joint probability $P(s,x)$ into table of conditional probabilies $P(s|x)$.
This can be done in two ways. A: Using product rule, $P(s|x) = P(s,x)/P(x)$.

First, to get $P(x)$, we use Sum rule (marginalizing).

	$x =$ *mild*	$x =$*irritated*	$x =$ *upset*	$x =$ *alarming*
$P(x)$	0.39	0.40	0.16	0.05

Second, applying product rule, $P(s|x) = P(s,x)/P(x)$.
B: calculating the probability on a "per column basis".
E.g. for the first cell, A: $0.35/0.39 = 0.897$ B: $0.35/(0.35+0.04)$

$P(s|x)$	$x =$ *mild*	$x =$*irritated*	$x =$ *upset*	$x =$ *alarming*
$s =$ *good*	0.897	0.7	0.438	0.00
$s =$ *average*	0.103	0.25	0.25	0.4
$s =$ *bad*	0.00	0.125	0.313	0.6

Having the table of all $P(s|x)$ we just mechanically insert into the equation in the slide title.

Statistical decision making: wrapping up

- **Given**:
  - A set of possible states : $\mathcal{S}$
  - A set of possible decisions : $\mathcal{D}$
  - A loss function $l : \mathcal{D} \times \mathcal{S} \to \Re$
  - The range $\mathcal{X}$ of the attribute
  - Distribution $P(x, s)$, $x \in \mathcal{X}, s \in \mathcal{S}$.
- **Define**:
  - Strategy : function $\delta : \mathcal{X} \to \mathcal{D}$
  - Risk of strategy δ : $r(\delta) = \sum_x \sum_s l(s, \delta(x)) P(x, s)$
- **Bayes problem**:
  - Goal: find the optimal strategy $\delta^* = \arg\min_\delta r(\delta)$
  - Solution: $\delta^*(x) = \arg\min_d \sum_s l(s, d) P(s|x)$ (for each x)

Notes

A special case - Bayesian *classification*

- Bayesian classification is a special case of statistical decision theory:
  - Attribute vector $\vec{x} = (x_1, x_2, \ldots)$: pixels 1, 2, ...
  - **State set $\mathcal{S}$ = decision set $\mathcal{D} = \{0, 1, \ldots 9\}$.**
  - State = actual class, Decision = recognized class
  - Loss function:

$$l(s,d) = \begin{cases} 0, & d = s \\ 1, & d \neq s \end{cases}$$

$$\delta^*(\vec{x}) = \arg\min_d \sum_s \underbrace{l(s,d)}_{0 \text{ if } d=s} P(s|\vec{x}) = \arg\min_d \sum_{s \neq d} P(s|\vec{x})$$

Obviously $\sum_s P(s|\vec{x}) = 1$, then:

$$P(d|\vec{x}) + \sum_{s \neq d} P(s|\vec{x}) = 1$$

Inserting into above:

$$\delta^*(\vec{x}) = \arg\min_d [1 - P(d|\vec{x})] = \arg\max_d P(d|\vec{x})$$

Notes

- Classification as opposed to Decision
- Loss function simply counts errors (misclassifications)
- We consider all errors equally painful!
- More examples during the lab ...
- The final result is not that surprising, is it? (Is it good or bad?)

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References I

Further reading: Chapter 13 and 14 of [6] (Chapters 12 and 13 in [7]). Books [1] (for this lecture, read Chapter 1) and [2] are classical textbooks in the field of pattern recognition and machine learning. Interesting insights into how people think and interact with probabilities are presented in [4] (in Czech as [5]).

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[6] Stuart Russell and Peter Norvig.
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Notes