Probabilistic Planning

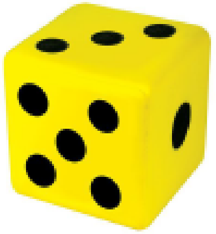
Stefan Edelkamp

PUI - CTU

Motivation

- Situations where actions have multiple possible outcomes and each outcome has a probability

- Several possible action representations
  - Bayes nets, probabilistic actions, …
- Book doesn't commit to any representation
  - Mainly concentrates on the underlying semantics



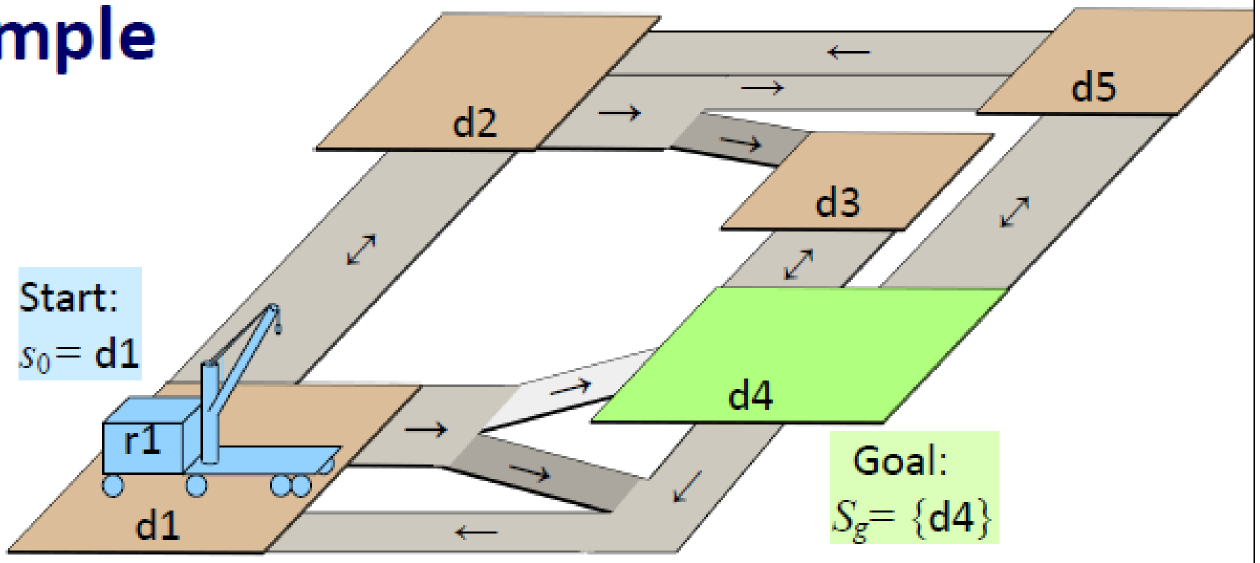
roll-die(d)
pre: holding(d) = true
eff:
1/6: top(d) $\leftarrow$ 1
1/6: top(d) $\leftarrow$ 2
1/6: top(d) $\leftarrow$ 3
1/6: top(d) $\leftarrow$ 4
1/6: top(d) $\leftarrow$ 5
1/6: top(d) $\leftarrow$ 6

Probabilistic Planning Domain

- $\Sigma = (S, A, \gamma, \Pr, \text{cost})$
  - S = set of states
  - A = set of actions
  - $\gamma : S \times A \rightarrow 2^S$
  - $\Pr(s' \mid s, a)$ = probability of going to state s' if we perform a in s
    - Require $\Pr(s' \mid s, a) \neq 0$ iff $s' \in \gamma(s,a)$
  - cost: $S \times A \rightarrow \mathbb{R}_{>0}$
    - $\text{cost}(s,a)$ = cost of action a in state s
    - may omit, default is $\text{cost}(s,a) = 1$

Example

- Robot r1 starts at d1
- Objective:
  get to d4



- Simplified
  state names:
  write {loc(r1)=d2} as d2
- Simplified action names:
  write move(r1,d2,d3) as m23
- r1 has unreliable steering,
  especially on hills
  - may slip and go elsewhere

m12: Pr(d2 | d1,m12) = 1

m21, m34, m41, m43, m45, m52, m54: like above

m14: Pr(d4 | d1,m14) = 0.5

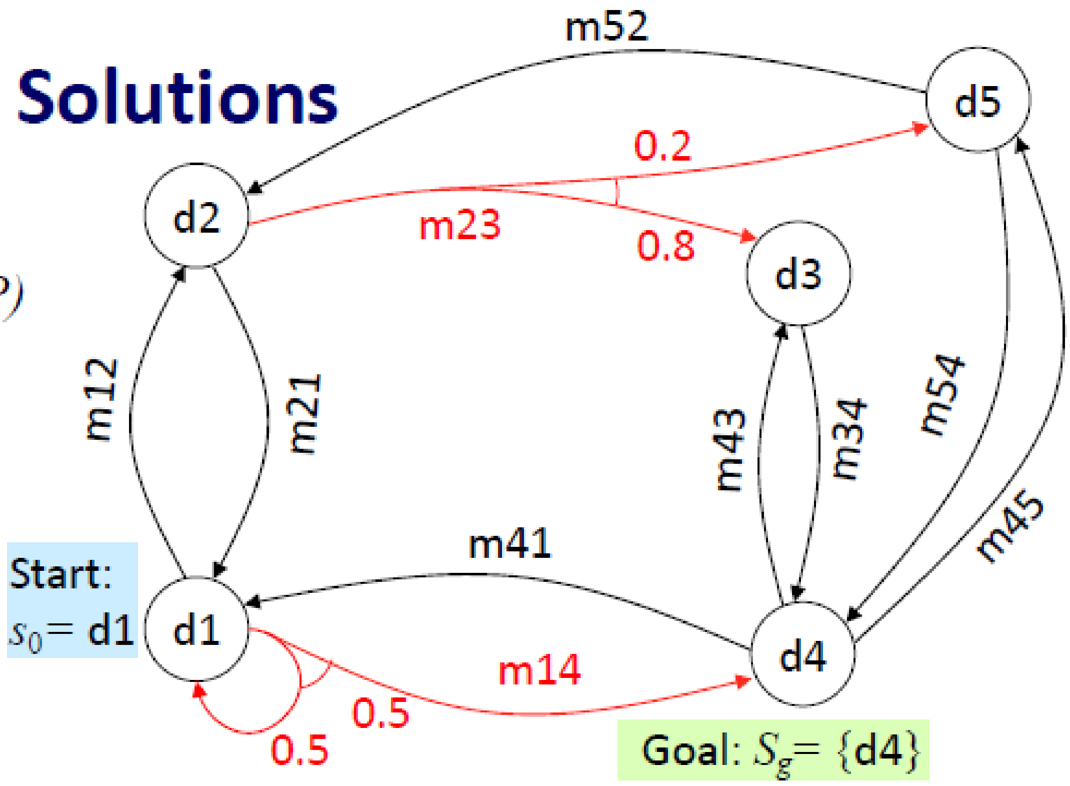
Pr(d1 | d1,m14) = 0.5

m23: Pr(d3 | d1,m23) = 0.8

Pr(d5 | d1,m23) = 0.2

Policies, Problems, Solutions

- *Stochastic shortest path (SSP)* problem:
  - a triple (Σ, s_0, S_g)
- *Policy*: partial function $\pi : S \rightarrow A$ such that for every $s \in \mathrm{Dom}(\pi) \subseteq S$, $\pi(s) \in \mathrm{Applicable}(s)$
- *Solution* for (Σ, s_0, S_g): a policy π such that $s_0 \in \mathrm{Dom}(\pi)$ and
  - ~~*leaves*~~$(s_0,\pi) \cap S_g \neq \emptyset$
  - $\hat{\gamma}(s_0,\pi) \cap S_g \neq \emptyset$



m14: $\Pr(d4 \mid d1, m14) = 0.5$

$\Pr(d1 \mid d1, m14) = 0.5$

m23: $\Pr(d3 \mid d1, m23) = 0.8$

$\Pr(d5 \mid d1, m23) = 0.2$

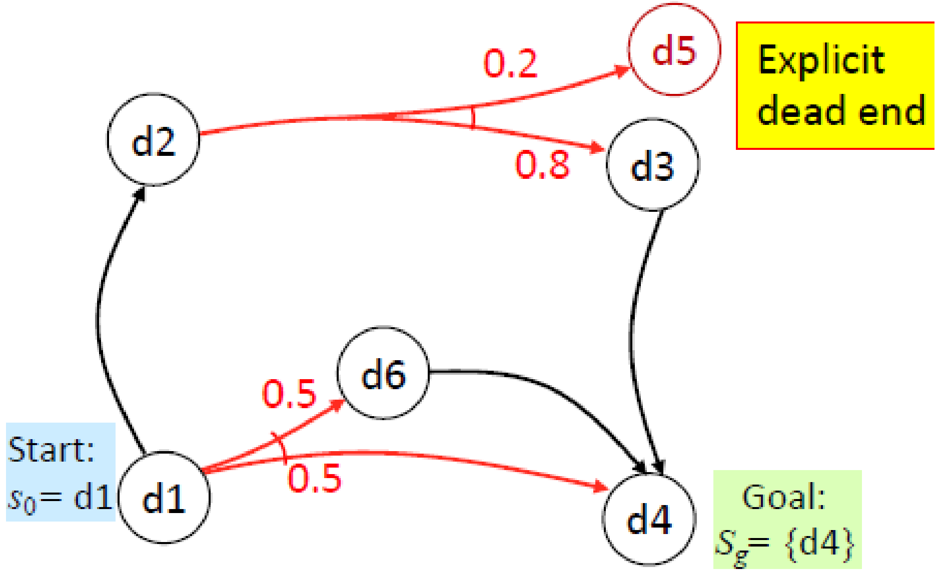
Notation and Terminology

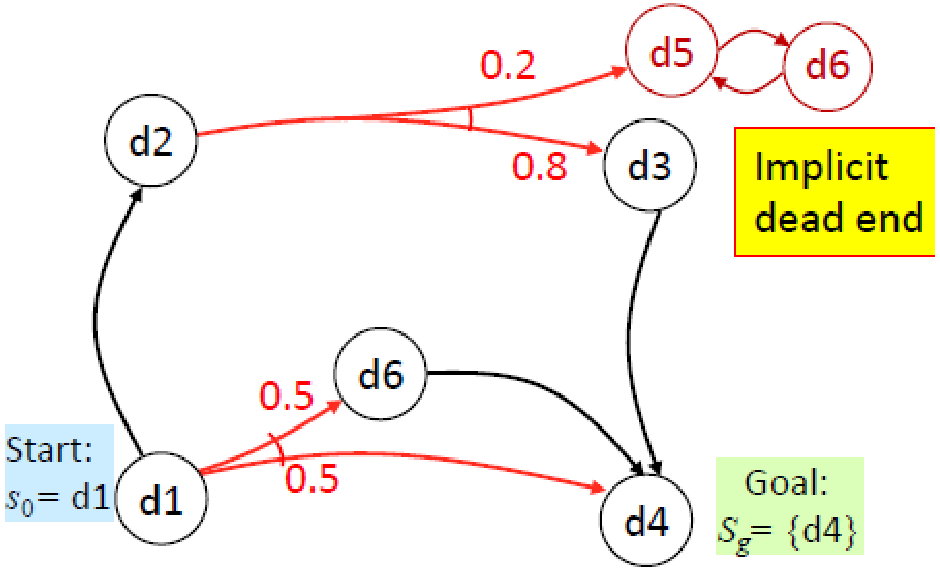
- *Transitive closure*
  - $\hat{\gamma}(s,\pi) = \{s$ and all states reachable from s using $\pi\}$
- Graph(s,π) = rooted graph induced by π at s
  - nodes: $\hat{\gamma}(s,\pi)$
  - edges: state transitions
- *leaves*$(s,\pi) = \hat{\gamma}(s,\pi) \setminus \mathrm{Dom}(\pi)$

- A solution policy π is *closed* if it doesn't stop at non-goal states unless there's no way to continue
  - for every state in $\hat{\gamma}(s,\pi)$, either
    - $s \in \mathrm{Dom}(\pi)$ (i.e., π specifies an action at s)
    - $s \in S_g$ (i.e., s is a goal state)
    - Applicable(s) = Ø (i.e., there are no applicable actions at s)

Dead Ends

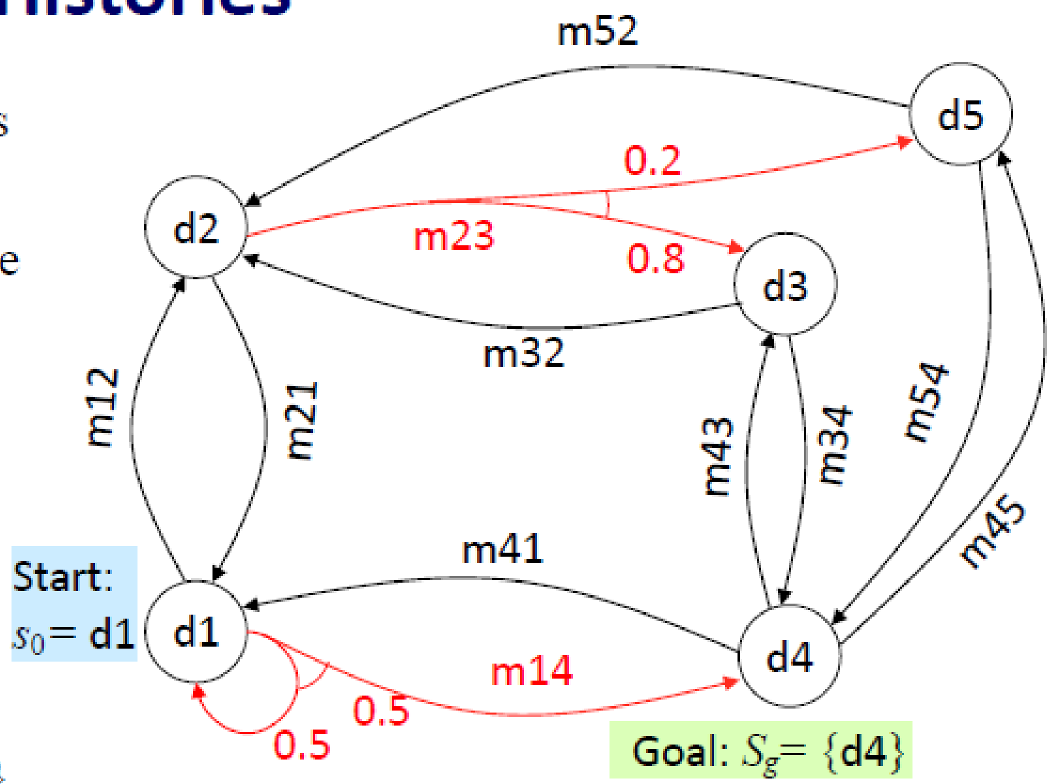
- Dead end:
  - A state or set of states from which the goal is unreachable





Histories

- **History**: sequence of states
  $\sigma = \langle s_0, s_1, s_2, \ldots \rangle$
  - May be finite or infinite
    $\sigma = \langle$ d1, d2, d3, d4$\rangle$
    $\sigma = \langle$d1, d2, d1, d2, …$\rangle$
- Let $H(s,\pi)$ = {all possible histories if we start at s and follow π, stopping if $\pi(s)$ is undefined *or if we reach a goal state*}

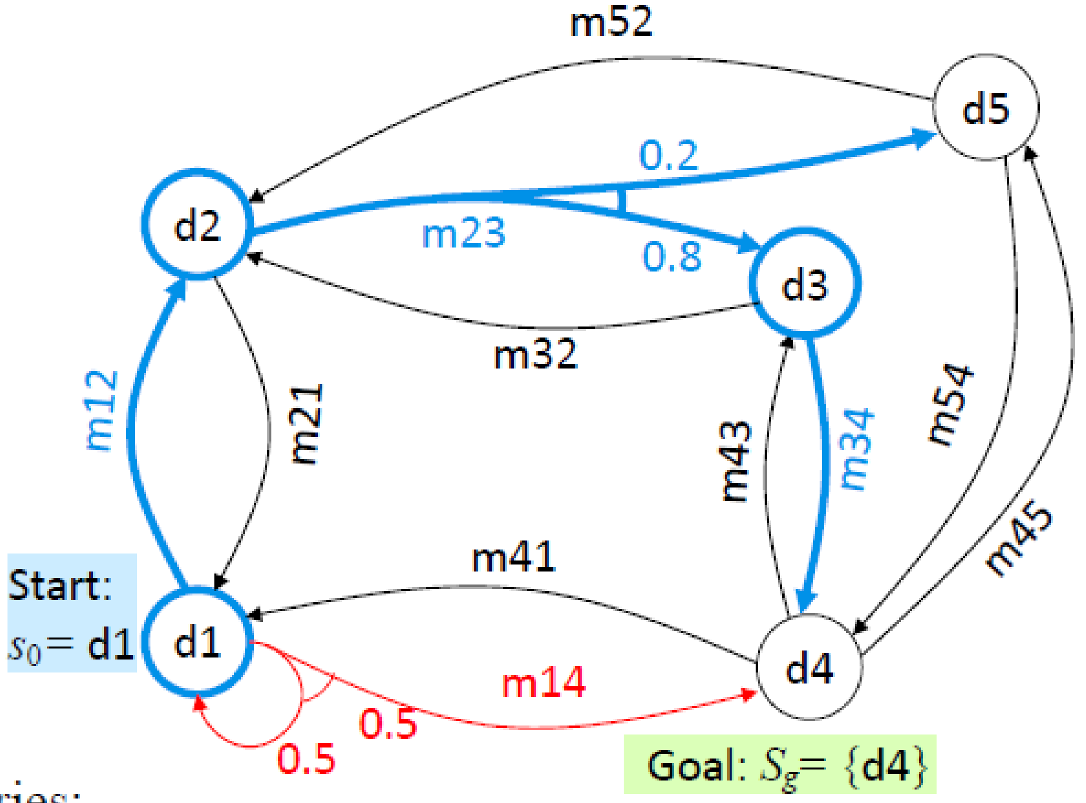


- If $\sigma \in H(s,\pi)$ then $\Pr(\sigma \mid s,\pi) = \prod_i \Pr(s_{i+1} \mid s_i, \pi(s_i))$
  - Thus $\sum_{\sigma \in H(s,\pi)} \Pr(\sigma \mid s,\pi) = 1$
- Probability of reaching a goal state:
  - $\Pr(S_g \mid s, \pi) = \sum_{\sigma \in H(s,\pi)} \{\Pr(\sigma \mid s,\pi) \mid \sigma$ ends at a state in $S_g\}$

Unsafe Solutions

- Unsafe solution:
  - $0 < \Pr(S_g \mid s_0, \pi) < 1$
- Example:

$\pi_1 = \{$(d1, m12),
(d2, m23),
(d3, m34)$\}$

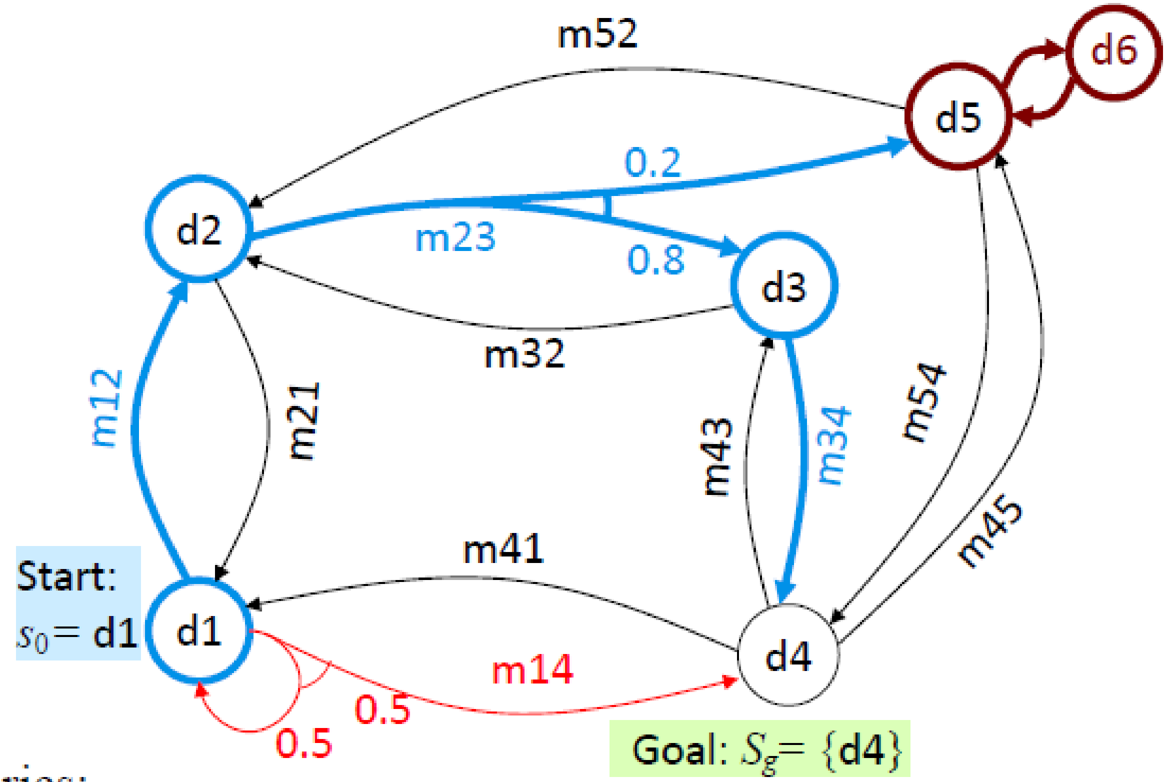


- $H(s_0, \pi_1)$ contains two histories:
  - $\sigma_1 = \langle$d1, d2, d3, d4$\rangle$ $\quad \Pr(\sigma_1 \mid s_0, \pi_1) = 1 \times .8 \times 1 = .8$
  - $\sigma_2 = \langle$d1, d2, d5$\rangle$ $\quad \Pr(\sigma_2 \mid s_0, \pi_1) = 1 \times .2 = .2$
- $\Pr(S_g \mid s_0, \pi_1) = .8$

Unsafe Solutions

- Unsafe solution:
  - $0 < \Pr(S_g \mid s_0, \pi) < 1$
- Example:

$\pi_2 = \{$(d1, m12),
(d2, m23),
(d3, m34),
(d5, move(r1,d5,d6)),
(d6, move(r1,d6,d5))$\}$

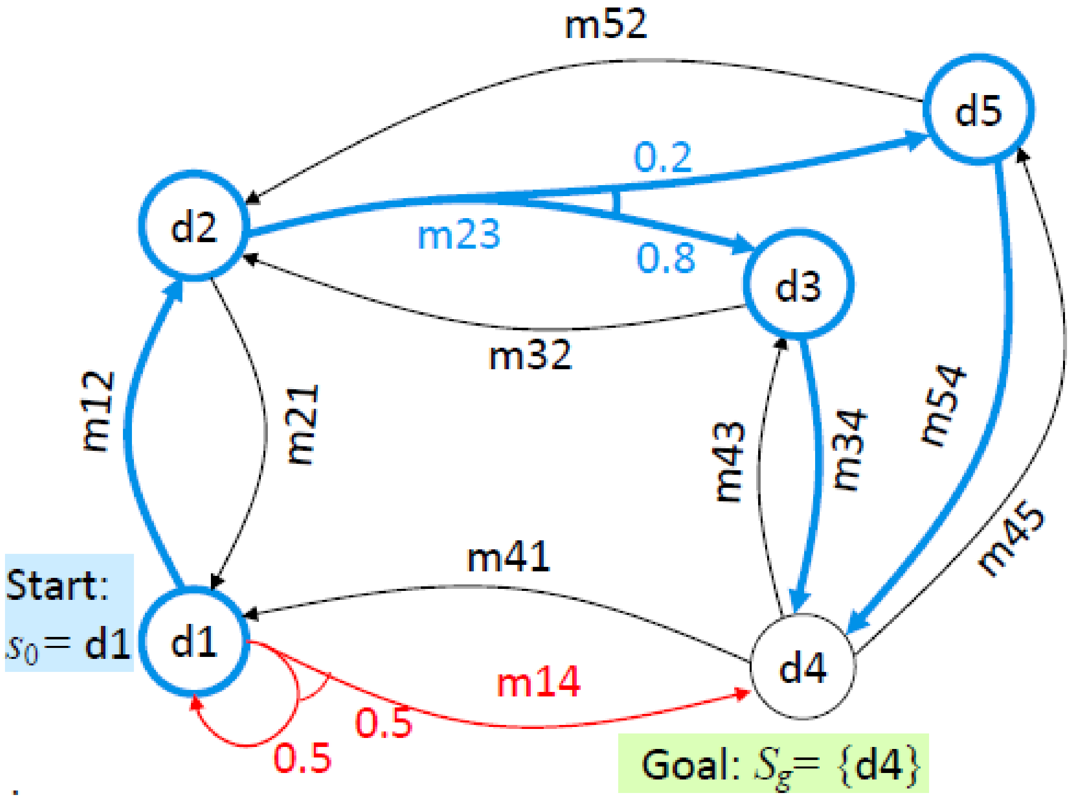


- $H(s_0, \pi_2)$ contains two histories:
  - $\sigma_1 = \langle$d1, d2, d3, d4$\rangle$ $\qquad \Pr(\sigma_1 \mid s_0, \pi_2) = 1 \times .8 \times 1 = .8$
  - $\sigma_3 = \langle$d1, d2, d5, d6, d5, d6, … $\rangle$ $\qquad \Pr(\sigma_3 \mid s_0, \pi_2) = 1 \times .2 \times 1 \times 1 \times 1 \times \ldots = .2$
- $\Pr(S_g \mid s_0, \pi_2) = .8$

Safe Solutions

- Safe solution:
  - $\Pr(S_g \mid s_0, \pi) = 1$
- An acyclic safe solution:

$\pi_3 = \{$(d1, m12),
(d2, m23),
(d3, m34),
(d5, m54)$\}$



- $H(s_0, \pi_3)$ contains two histories:
  - $\sigma_1 = \langle$d1, d2, d3, d4$\rangle$ $\quad \Pr(\sigma_1 \mid s_0, \pi_3) = 1 \times .8 \times 1 = .8$
  - $\sigma_4 = \langle$d1, d2, d5, d4$\rangle$ $\quad \Pr(\sigma_4 \mid s_0, \pi_3) = 1 \times .2 \times 1 = .2$

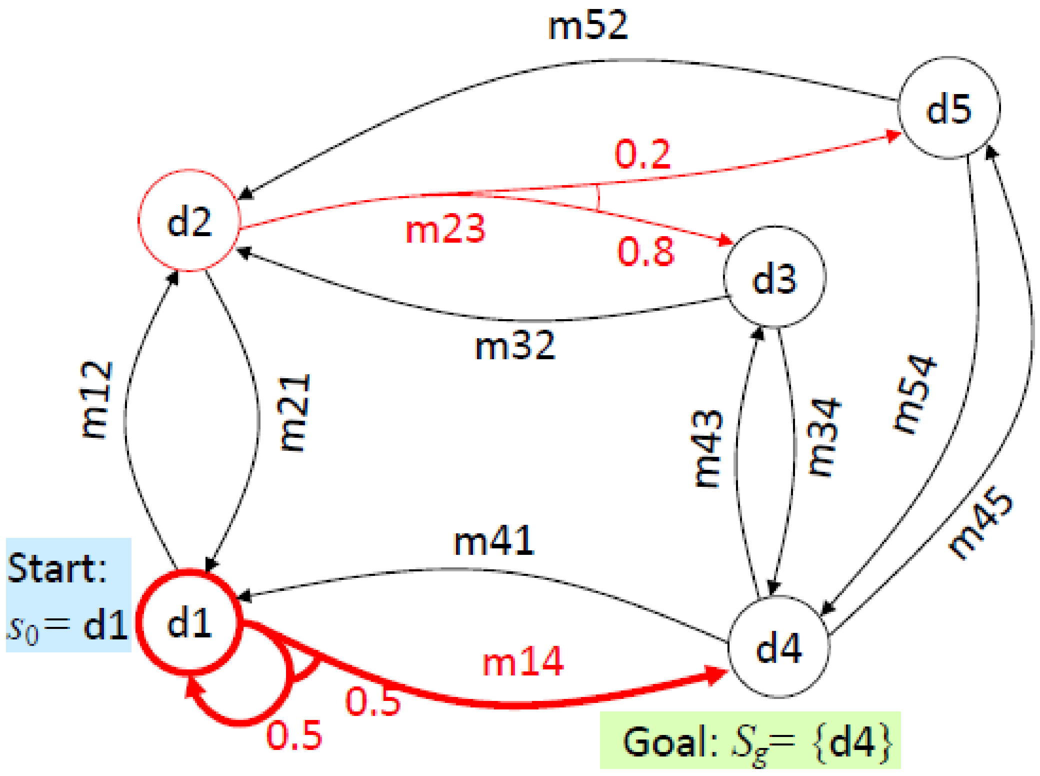
$\Pr(S_g \mid s_0, \pi_3) = .8 + .2 = 1$

Safe Solutions

- Safe solution:
  - $\Pr(S_g \mid s_0, \pi) = 1$
- A cyclic safe solution:
  π_4 = {(d1, m54)}
- $H(\pi_4)$ contains infinitely many histories:
  - $\sigma_5 = \langle$d1, d4$\rangle$ $\Pr(\sigma_5 \mid s_0, \pi_4) = ½$
  - $\sigma_6 = \langle$d1, d1, d4$\rangle$ $\Pr(\sigma_6 \mid s_0, \pi_4) = (½)^2 = ¼$
  - $\sigma_7 = \langle$d1, d1, d1, d4$\rangle$ $\Pr(\sigma_6 \mid s_0, \pi_4) = (½)^3 = {}^1/_8$

  …

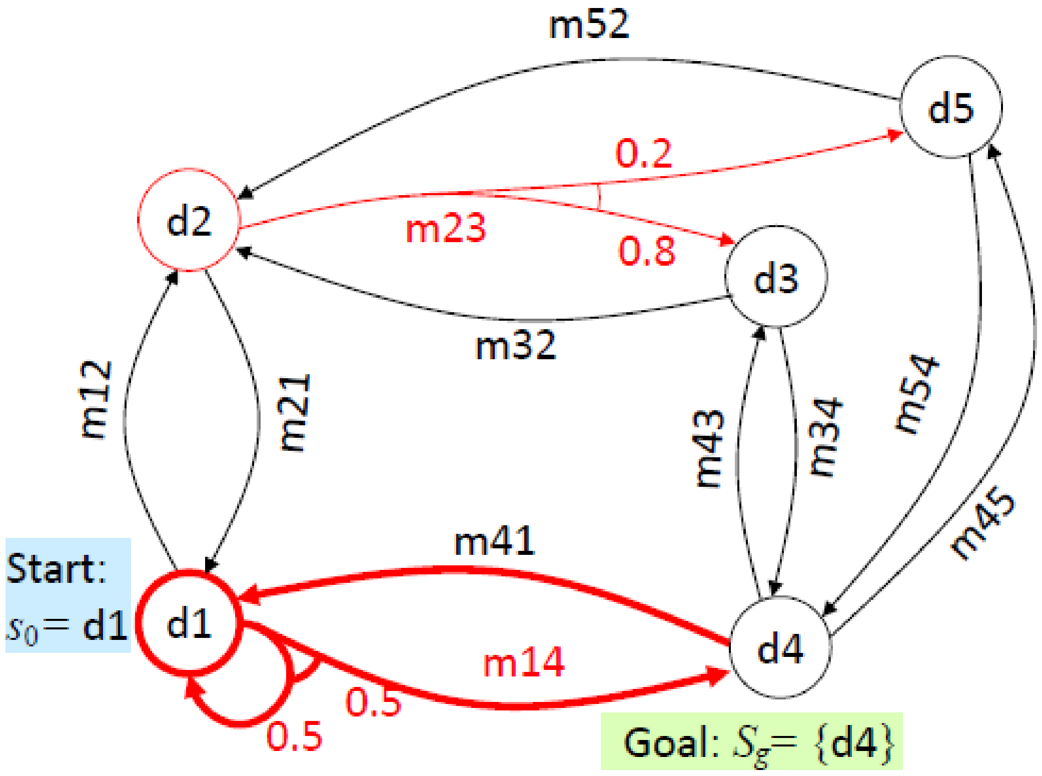
$\Pr(S_g \mid s_0, \pi_4) = ½ + ¼ + {}^1/_8 + \ldots = 1$



Safe Solutions

- Safe solution:
  - $\Pr(S_g \mid s_0, \pi) = 1$
- Another cyclic safe solution:

  π_5 = {(d1, m54), (d4, m41)}

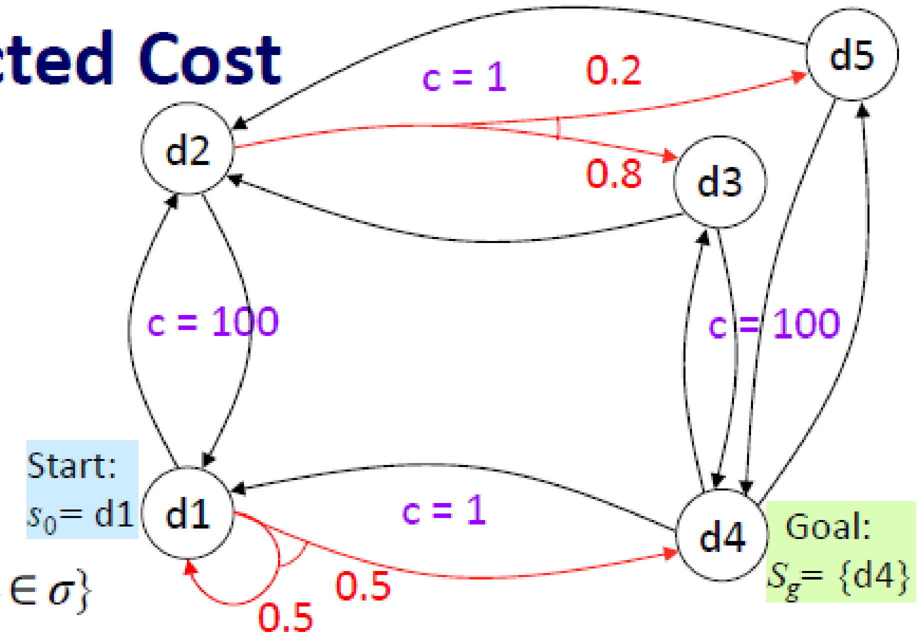


- $H(\pi_5) = H(\pi_4)$:
  - $\sigma_5 = \langle$d1, d4$\rangle$ $\qquad \Pr(\sigma_5 \mid s_0, \pi_4) = ½$
  - $\sigma_6 = \langle$d1, d1, d4$\rangle$ $\qquad \Pr(\sigma_6 \mid s_0, \pi_4) = (½)^2 = ¼$
  - $\sigma_7 = \langle$d1, d1, d1, d4$\rangle$ $\qquad \Pr(\sigma_6 \mid s_0, \pi_4) = (½)^3 = 1/8$

  ...

$\Pr(S_g \mid s_0, \pi_4) = ½ + ¼ + 1/8 + \ldots = 1$

Expected Cost

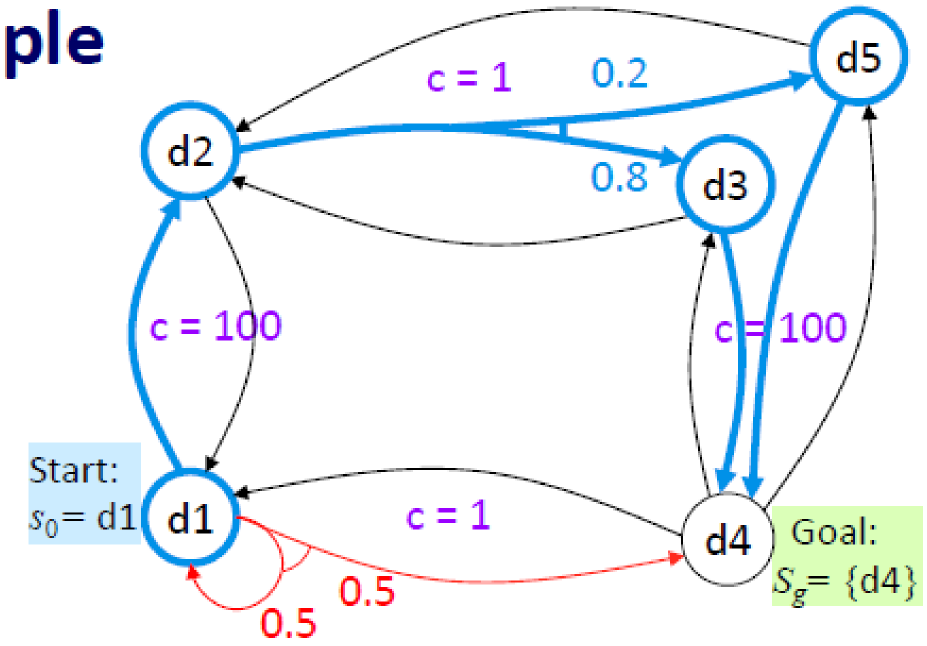


- $\text{cost}(s,a)$ = cost of using a in s
- Example:
  - each "horizontal" action costs 1
  - each "vertical" action costs 100
- History $\sigma = \langle s_0, s_1, s_2, \ldots \rangle$
  - $\text{cost}(\sigma \mid s_0, \pi) = \sum\{\text{cost}(s_i, \pi(s_i)) \mid s_i \in \sigma\}$
- Let π be a safe solution
- At each state $s \in \text{Dom}(\pi)$, expected cost of following π to goal:
  - Weighted sum of history costs:
    - $V^\pi(s) = \text{cost}(s,\pi(s)) + \sum_{\sigma \in H(s,\pi)} \Pr(\sigma \mid s,\pi)\, \text{cost}(\sigma \mid s,\pi)$
  - Recursive equation

$$V^\pi(s) = \begin{cases} 0, & \text{if } s \text{ is a goal} \\ \text{cost}(s,\pi(s)) + \sum_{s' \in \gamma(s,\pi(s))} \Pr(s' \mid s, \pi(s))\, V^\pi(s'), & \text{otherwise} \end{cases}$$

Example

- $\pi_3 = \{$(d1, m12),
  (d2, m23),
  (d3, m34),
  (d5, m54)$\}$



- Weighted sum of history costs:
  - $\sigma_1 = \langle$d1, d2, d3, d4$\rangle$
    - $\Pr(\sigma_1 \mid s_0, \pi_3) = 0.8$
    - $\text{cost}(\sigma_1 \mid s_0, \pi_3)$
      $= 100 + 1 + 100 = 201$
  - $\sigma_2 = \langle$d1, d2, d5, d4$\rangle$
    - $\Pr(\sigma_2 \mid s_0, \pi_3) = 0.2$
    - $\text{cost}(\sigma_2 \mid s_0, \pi_3)$
      $= 100 + 1 + 100 = 201$

- Recursive equation:

$$
\begin{aligned}
V^{\pi_1}(\text{d1}) &= 100 + V^{\pi_1}(\text{d2}) \\
&= 100 + 1 + .8V^{\pi_1}(\text{d3}) + .2V^{\pi_1}(\text{d5}) \\
&= 100 + 1 + .8(100) + .2(100) \\
&= 201
\end{aligned}
$$

- $V^{\pi_1}(\text{d1}) = .8(201) + .2(201) = 201$

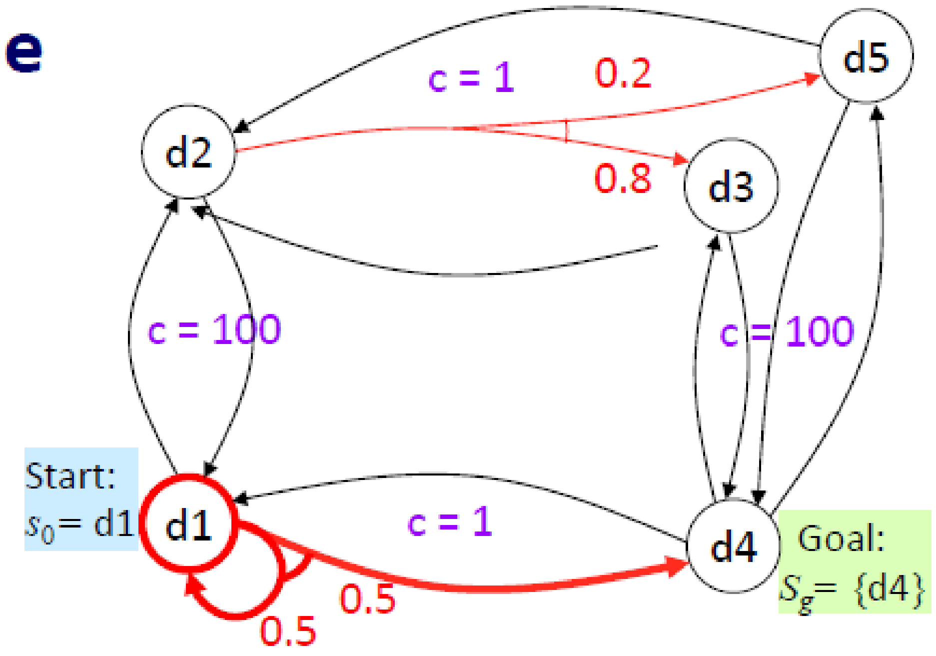
Example

- π_4 = {(d5, m54}

- Weighted sum of history costs:
  - $\sigma_5 = \langle$d1, d4$\rangle$
    - $\Pr(\sigma_5 \mid \pi_4) = ½$
    - $\text{cost}(\sigma_5 \mid \pi_4) = 1$
  - $\sigma_6 = \langle$d1, d1, d4$\rangle$
    - $\Pr(\sigma_6 \mid \pi_4) = (½)^2$
    - $\text{cost}(\sigma_6 \mid \pi_4) = 2$
  - $\sigma_7 = \langle$d1, d1, d1, d4$\rangle$
    - $\Pr(\sigma_7 \mid \pi_4) = (½)^3$
    - $\text{cost}(\sigma_7 \mid \pi_4) = 3$

  …

- $V^{\pi_4}(\text{d1}) = (½)1 + (½)^2\,2 + (½)^3\,3 + \ldots$
  $= 2$

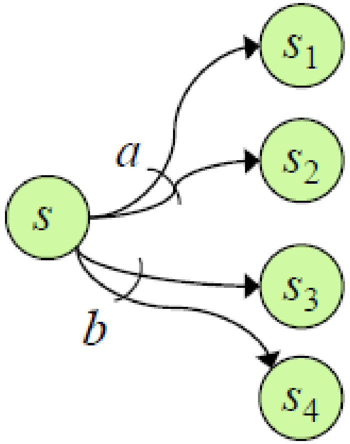


- Recursive equation:

$$V^{\pi_4}(\text{d1}) = 1 + ½(0) + ½(V^{\pi_4}(\text{d1}))$$
$$½V^{\pi_4}(\text{d1}) = 1$$
$$V^{\pi_4}(\text{d1}) = 2$$

Planning as Optimization

- Let π and π' be safe solutions
  - π *dominates* π' if $V^\pi(s) \le V^{\pi'}(s)$ for every $s \in \mathrm{Dom}(\pi) \cap \mathrm{Dom}(\pi')$
- π is *optimal* if π dominates *every* safe solution
  - If π and π' are both optimal, then $V^\pi(s) = V^{\pi'}(s)$ at every state where they're both defined
- $V^*(s)$ = expected cost of getting to goal using an optimal safe solution



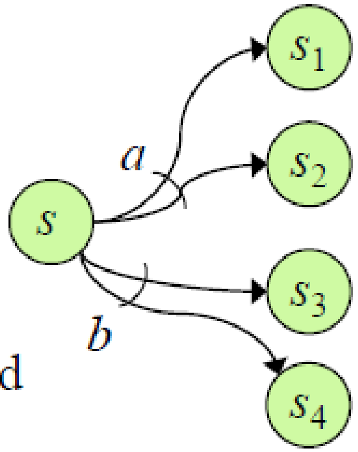
- Recall that $V^\pi(s) = \begin{cases} 0, & \text{if } s \text{ is a goal} \\ \mathrm{cost}(s,\pi(s)) + \sum_{s' \in \gamma(s,\pi(s))} \Pr(s' \mid s, \pi(s))\, V^\pi(s'), & \text{otherwise} \end{cases}$
- *Optimality principle* (Bellman's theorem):

$$V^*(s) = \begin{cases} 0, & \text{if } s \text{ is a goal} \\ \min_{a \in \mathrm{Applicable}(s)} \{\mathrm{cost}(s,a) + \sum_{s' \in \gamma(s,a)} \Pr(s' \mid s, a)\, V^*(s'), & \text{otherwise} \end{cases}$$

- Intuition: consider what would happen if $V^*(s) \ne \min_{a \in \mathrm{Applicable}(s)} \{\ldots\}$

Cost to Go

- Let (Σ, s_0, S_g) be a *safe* SSP
  - i.e., S_g is reachable from every state
  - same as *safely explorable* in Chapter 5
- Let π be a safe solution that's defined at all non-goal states
  - i.e., $\mathrm{Dom}(\pi) = S \setminus S_g$
- Let $a \in \mathrm{Applicable}(s)$
- *Cost-to-go*:
  - Expected cost if we start at s, use a, and use π afterward
  - $Q^\pi(s,a) = \mathrm{cost}(s,a) + \sum_{s' \in \gamma(s,a)} \Pr(s' \mid s,a)\, V^\pi(s')$
- For every $s \in S \setminus S_g$, let $\pi'(s) \in \mathrm{argmin}_{a \in \mathrm{Applicable}(s)}\, Q^\pi(s,a)$



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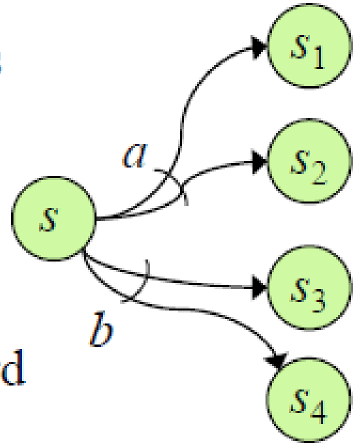
Cost to Go

- Let (Σ, s_0, S_g) be a *safe* SSP
  - i.e., S_g is reachable from every state

- Let π be a safe solution that's defined at all non-goal states
  - i.e., $\mathrm{Dom}(\pi) = S \setminus S_g$
- Let $a \in \mathrm{Applicable}(s)$
- *Cost-to-go*:
  - Expected cost if we start at s, use a, and use π afterward
  - $Q^\pi(s,a) = \mathrm{cost}(s,a) + \sum_{s' \in \gamma(s,a)} \Pr(s' \mid s,a)\, V^\pi(s')$



- For every $s \in S \setminus S_g$, let $\pi'(s) \in \mathrm{argmin}_{a \in \mathrm{Applicable}(s)}\, Q^\pi(s,a)$

Policy Iteration

- PI(Σ, s_0, S_g, π_0)

  $\pi \leftarrow \pi_0$

  loop

  compute $\{V^\pi(s) \mid s \in S\}$

  for every non-goal state s do

  $A \leftarrow \text{argmin}_{a \in \text{Applicable}(s)} Q^\pi(s,a)$

  if $\pi(s) \in A$ then $\pi'(s) \leftarrow \pi(s)$

  else $\pi'(s) \leftarrow$ any action in A

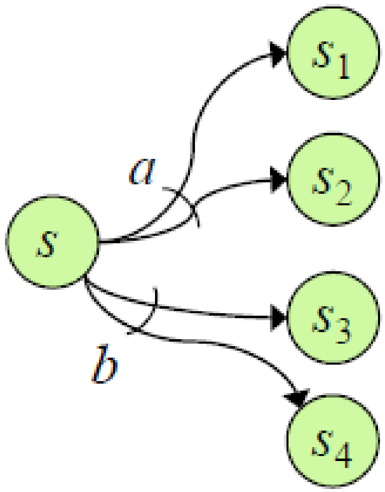
  if $\pi' = \pi$ then

  return π

  $\pi \leftarrow \pi'$

n equations, n unknowns, where $n = |S|$

E(cost of using a then π)

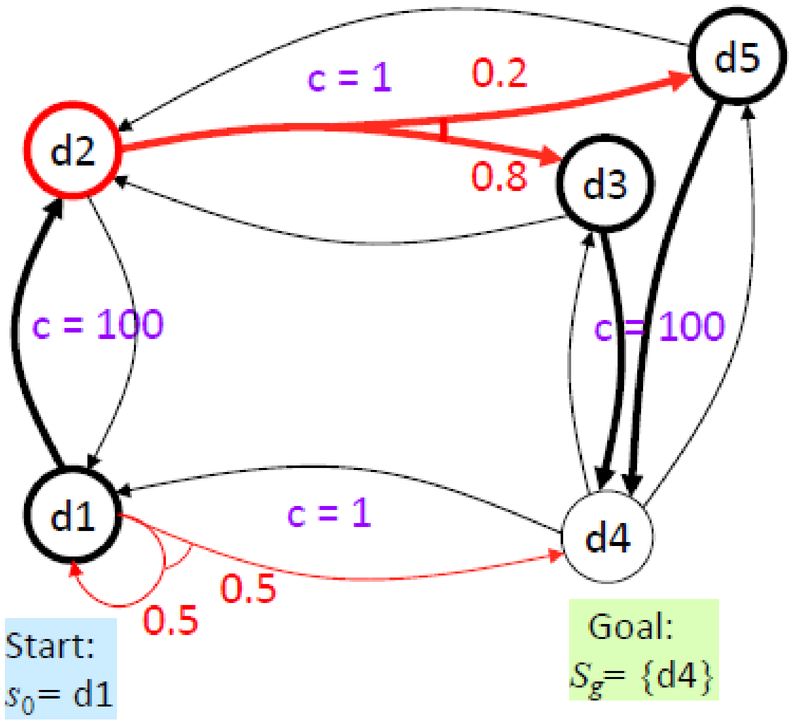


- Converges in a finite number of iterations

Example

Start with

$\pi = \pi_0 = \{$(d1, m12),
(d2, m23),
(d3, m34),
(d5, m54)$\}$



V^π(d4) = 0

V^π(d3) = 100 + V^π(d4) = 100

V^π(d5) = 100 + V^π(d5) = 100

V^π(d2) = 1 + (0.8 V^π(d3) + 0.2 V^π(d5)) = 101

V^π(d1) = 100 + V^π(d2) = 201

Q(d1,m12) = 100 + 101 = 201

Q(d1,m14) = 1 + ½ × 201 + ½(0) = 101.5

argmin = m14

Q(d2,m23) = 1 + (0.8(100) + 0.2(100)) = 101

Q(d2,m21) = 100 + 201 = 301

argmin = m23

Q(d3,m34) = 100 + 0 = 100

Q(d3,move(r1,d3,d2)) = 100 + 101 = 201

argmin = m34

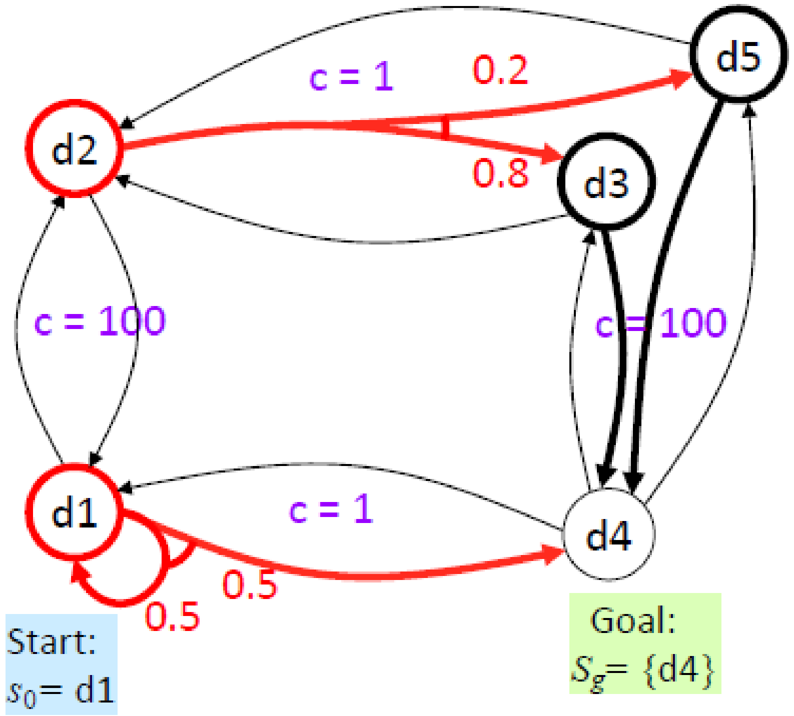
Q(d5,m54) = 100 + 0 = 100

Q(d5,m54) = 100+101 = 201

argmin = m54

Example

$\pi = \{$(d1, m14),
(d2, m23),
(d3, m34),
(d5, m54)$\}$



V^π(d4) = 0
V^π(d3) = 100 + V^π(d4) = 100
V^π(d5) = 100 + V^π(d5) = 100
V^π(d2) = 1 + (0.8 V^π(d3) + 0.2 V^π(d5)) = 101
V^π(d1) = 1 + ½V^π(d1) + ½V^π(d4) $\Rightarrow$ V^π(d1) = 2

Q(d1,m12) = 100 + 101 = 201
Q(d1,m14) = 1 + ½(2) + ½(0) = 2
argmin = m14

Q(d2,m23) = 1 + (0.8(100) + 0.2(100)) = 101
Q(d2,m21) = 100 + 2 = 102
argmin = m23

Q(d3,m34) = 100 + 0 = 100
Q(d3,move(r1,d3,d2)) = 100 + 101 = 201
argmin = m34

Q(d5,m54) = 100 + 0 = 100
Q(d5,m54) = 100+101 = 201
argmin = m54

Value Iteration

- Synchronous version (easier to understand)

$\text{VI}(\Sigma, s_0, S_g, V_0)$
  for $i = 1, 2, \ldots$
    for every nongoal state s
      for every applicable action a do
        $Q(s,a) \leftarrow \text{cost}(s,a) + \sum_{s' \in S} \Pr(s'|s,a) V_{i-1}(s')$
      $V_i(s) \leftarrow \min_{a \in \text{Applicable}(s)} Q(s,a)$
      $\pi_i(s) \leftarrow \text{argmin}_{a \in \text{Applicable}(s)} Q(s,a)$
    if $\max_{s \in S} |V_i(s) - V_{i-1}(s)| \leq \eta$ then return π'

- $\eta > 0$: for testing approximate convergence
- V_0 is a heuristic function
  - must have $V_0(s) = 0$ for every $s \in S_g$
  - e.g., adapt a heuristic from Chapter 2
- V_i = values computed at i'th iteration
- π_i = plan computed from V_i

- Asynchronous version (more efficient)

$\text{VI}(\Sigma, s_0, S_g, V_0)$
  global $\pi \leftarrow \emptyset$; global $V(s) \leftarrow V_0(s)\ \forall s$
  loop
    $r \leftarrow \max_{s \in S \setminus S_g}$ Bellman-Update(s)
    if $r \leq \eta$ then return π

Bellman-Update(s)
  $v_{\text{old}} \leftarrow V(s)$
  for every $a \in \text{Applicable}(s)$ do
    $Q(s,a) \leftarrow \text{cost}(s,a) + \sum_{s' \in S} \Pr(s'|s,a)\, V(s')$
  $V(s) \leftarrow \min_{a \in \text{Applicable}(s)} Q(s,a)$
  $\pi(s) \leftarrow \text{argmin}_{a \in \text{Applicable}(s)} Q(s,a)$
  return $|V(s) - v_{\text{old}}|$

- Synchronous version computes V_i and π_i from old V_{i-1} and π_{i-1}
- Asynchronous version updates V and π in place; new values available immediately

Synchronous

$Q(\text{d1},\text{m12}) = 100 + 0 = 100$

$Q(\text{d1},\text{m14})$
$= 1 + (½(0) + ½(0)) = 1$

$V_1(\text{d1}) = 1;\ \pi_1(\text{d1}) = \text{m14}$

$Q(\text{d2},\text{m21}) = 100 + 0 = 100$

$Q(\text{d2},\text{m23}) = 1 + (½(0) + ½(0)) = 1$

$V_1(\text{d2}) = 1;\ \pi_1(\text{d2}) = \text{m23}$

$Q(\text{d3},\text{m32}) = 1 + 0 = 1$

$Q(\text{d3},\text{m34}) = 100 + 0 = 100$

$V_1(\text{d3}) = 1;\ \pi_1(\text{d3}) = \text{m32}$

$Q(\text{d5},\text{m52}) = 1 + 0 = 1$

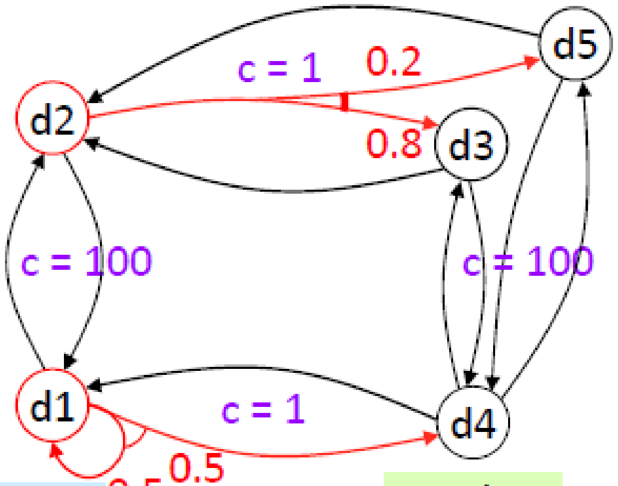
$Q(\text{d5},\text{m54}) = 100 + 0 = 100$

$V_1(\text{d5}) = 1;\ \pi_1(\text{d5}) = \text{m52}$

$r = \max(1 - 0, 1 - 0, 1 - 0, 1 - 0) = 1$

$\eta = 0.2$

$V_0(s) = 0$ for all s



Start: s_0 = d1

Goal: S_g = {d4}

Asynchronous

$Q(\text{d1},\text{m12}) = 100 + 0 = 100$

$Q(\text{d1},\text{m14})$
$= 1 + (½(0) + ½(0)) = 1$

$V(\text{d1}) = 1;\ \pi(\text{d1}) = \text{m14}$

$Q(\text{d2},\text{m21}) = 100 + 1 = 101$

$Q(\text{d2},\text{m23}) = 1 + (½(0) + ½(0)) = 1$

$V(\text{d2}) = 1;\ \pi(\text{d2}) = \text{m23}$

$Q(\text{d3},\text{m32}) = 1 + 1 = 2$

$Q(\text{d3},\text{m34}) = 100 + 0 = 100$

$V(\text{d3}) = 2;\ \pi(\text{d3}) = \text{m32}$

$Q(\text{d5},\text{m52}) = 1 + 1 = 2$

$Q(\text{d5},\text{m54}) = 100 + 0 = 100$

$V(\text{d5}) = 2;\ \pi(\text{d5}) = \text{m52}$

$r = \max(1 - 0,\ 1 - 0,\ 2 - 0,\ 2 - 0) = 1$

Synchronous

$Q(\text{d1,m12}) = 100 + 1 = 101$

$Q(\text{d1,m14})$
$= 1 + (½(1) + ½(0)) = 1½$

$V_2(\text{d1}) = 1½;\ \pi_2(\text{d1}) = \text{m14}$

$Q(\text{d2,m21}) = 100 + 1 = 101$

$Q(\text{d2,m23}) = 1 + (½(1) + ½(1)) = 2$

$V_2(\text{d2}) = 2;\ \pi_2(\text{d2}) = \text{m23}$

$Q(\text{d3,m32}) = 1 + 1 = 2$

$Q(\text{d3,m34}) = 100 + 0 = 100$

$V_2(\text{d3}) = 2;\ \pi_2(\text{d3}) = \text{m32}$

$Q(\text{d5,m52}) = 1 + 1 = 2$

$Q(\text{d5,m54}) = 100 + 0 = 100$

$V_2(\text{d5}) = 2;\ \pi_2(\text{d5}) = \text{m52}$

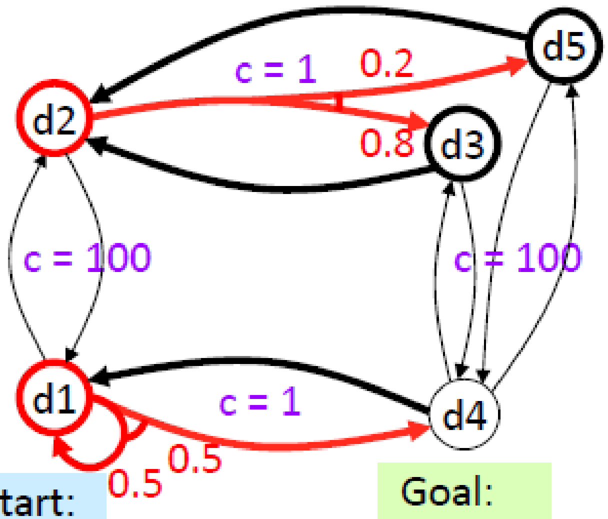
$r = \max(1½ - 1,\ 2 - 1,\ 2 - 1,\ 2 - 1) = 1$

$\eta = 0.2$

$V(\text{d1}) = 1$
$V(\text{d2}) = 1$
$V(\text{d3}) = 1$
$V(\text{d5}) = 1$

$\eta = 0.2$

$V(\text{d1}) = 1$
$V(\text{d2}) = 1$
$V(\text{d3}) = 2$
$V(\text{d5}) = 2$



Start:
s_0 = d1

Goal:
S_g = {d4}

Asynchronous

$Q(\text{d1,m12}) = 100 + 0 = 101$

$Q(\text{d1,m14})$
$= 1 + (½(1) + ½(0)) = 1½$

$V(\text{d1}) = 1;\ \pi(\text{d1}) = \text{m14}$

$Q(\text{d2,m21}) = 100 + 1½ = 101½$

$Q(\text{d2,m23}) = 1 + (½(2) + ½(2)) = 3$

$V(\text{d2}) = 3;\ \pi(\text{d2}) = \text{m23}$

$Q(\text{d3,m32}) = 1 + 3 = 4$

$Q(\text{d3,m34}) = 100 + 0 = 100$

$V(\text{d3}) = 4;\ \pi(\text{d3}) = \text{m32}$

$Q(\text{d5,m52}) = 1 + 3 = 4$

$Q(\text{d5,m54}) = 100 + 0 = 100$

$V(\text{d5}) = 4;\ \pi(\text{d5}) = \text{m52}$

$r = \max(1½ - 1,\ 3 - 1,\ 4 - 2,\ 4 - 2) = 2$

Synchronous

$\eta = 0.2$

$V(\text{d1}) = 1\frac{1}{2}$
$V(\text{d2}) = 2$
$V(\text{d3}) = 2$
$V(\text{d5}) = 2$

$Q(\text{d1,m12}) = 100 + 2 = 102$

$Q(\text{d1,m14}) = 1 + (\frac{1}{2}(1\frac{1}{2}) + \frac{1}{2}(0)) = 1\frac{3}{4}$

$V_3(\text{d1}) = 1\frac{3}{4};\ \pi_3(\text{d1}) = \text{m14}$

$Q(\text{d2,m21}) = 100 + 1\frac{1}{2} = 101\frac{1}{2}$

$Q(\text{d2,m23}) = 1 + (\frac{1}{2}(2) + \frac{1}{2}(2)) = 3$

$V_3(\text{d2}) = 3;\ \pi_3(\text{d2}) = \text{m23}$

$Q(\text{d3,m32}) = 1 + 2 = 3$

$Q(\text{d3,m34}) = 100 + 0 = 100$

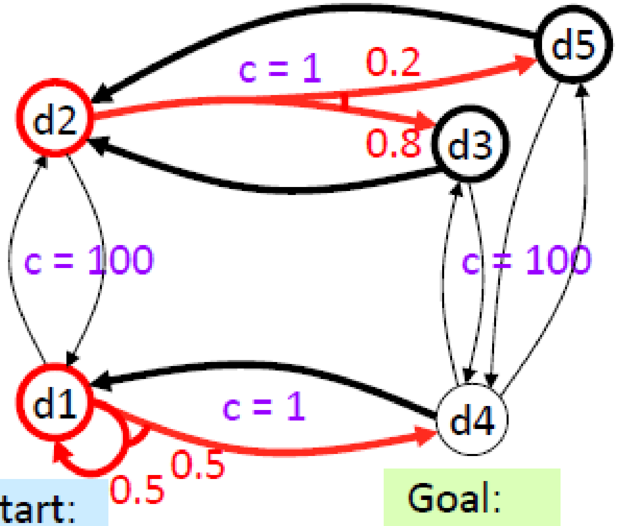
$V_3(\text{d3}) = 3;\ \pi_3(\text{d3}) = \text{m32}$

$Q(\text{d5,m52}) = 1 + 2 = 3$

$Q(\text{d5,m54}) = 100 + 0 = 100$

$V_3(\text{d5}) = 3;\ \pi_3(\text{d5}) = \text{m52}$

$r = \max(1\frac{3}{4} - 1\frac{1}{2},\ 3-2,\ 3-2,\ 3-2) = 1$



Start: s_0 = d1

Goal: S_g = {d4}

Asynchronous

$\eta = 0.2$

$V(\text{d1}) = 1\frac{1}{2}$
$V(\text{d2}) = 3$
$V(\text{d3}) = 4$
$V(\text{d5}) = 4$

$Q(\text{d1,m12}) = 100 + 3 = 103$

$Q(\text{d1,m14}) = 1 + (\frac{1}{2}(1\frac{1}{2}) + \frac{1}{2}(0)) = 1\frac{3}{4}$

$V(\text{d1}) = 1\frac{3}{4};\ \pi(\text{d1}) = \text{m14}$

$Q(\text{d2,m21}) = 100 + 1\frac{3}{4} = 101\frac{3}{4}$

$Q(\text{d2,m23}) = 1 + (\frac{1}{2}(4) + \frac{1}{2}(4)) = 5$

$V(\text{d2}) = 5;\ \pi(\text{d2}) = \text{m23}$

$Q(\text{d3,m32}) = 1 + 5 = 6$

$Q(\text{d3,m34}) = 100 + 0 = 100$

$V(\text{d3}) = 6;\ \pi(\text{d3}) = \text{m32}$

$Q(\text{d5,m52}) = 1 + 5 = 6$

$Q(\text{d5,m54}) = 100 + 0 = 100$

$V(\text{d5}) = 6;\ \pi(\text{d5}) = \text{m52}$

$r = \max(1\frac{3}{4} - 1\frac{1}{2},\ 5-3,\ 6-4,\ 6-4) = 2$

Synchronous

$Q(\text{d1},\text{m12}) = 100 + 0 = 100$

$Q(\text{d1},\text{m14})$
$= 1 + (\tfrac{1}{2}(1^3/_4) + \tfrac{1}{2}(0)) = 1^7/_8$

$V_4(\text{d1}) = 1^7/_8;\ \pi_4(\text{d1}) = \text{m14}$

$Q(\text{d2},\text{m21}) = 100 + 1^3/_4 = 101^3/_4$

$Q(\text{d2},\text{m23}) = 1 + (\tfrac{1}{2}(3) + \tfrac{1}{2}(3)) = 4$

$V_4(\text{d2}) = 4;\ \pi_4(\text{d2}) = \text{m23}$

$Q(\text{d3},\text{m32}) = 1 + 3 = 4$

$Q(\text{d3},\text{m34}) = 100 + 0 = 100$

$V_4(\text{d3}) = 4;\ \pi_4(\text{d3}) = \text{m32}$

$Q(\text{d5},\text{m52}) = 1 + 3 = 4$

$Q(\text{d5},\text{m54}) = 100 + 0 = 100$

$V_4(\text{d5}) = 4;\ \pi_4(\text{d5}) = \text{m52}$

$r = \max(1^7/_8 - 1^3/_4,\ 3 - 2,\ 3 - 2,\ 3 - 2) = 1$

$\eta = 0.2$

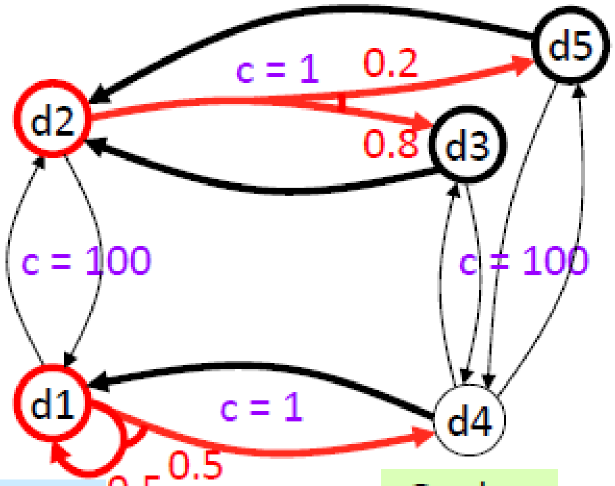
$V(\text{d1}) = 1^3/_4$
$V(\text{d2}) = 3$
$V(\text{d3}) = 3$
$V(\text{d5}) = 3$

$\eta = 0.2$

$V(\text{d1}) = 1^3/_4$
$V(\text{d2}) = 5$
$V(\text{d3}) = 6$
$V(\text{d5}) = 6$

How long before $r \le \eta$?

How long, if the "vertical" actions cost 10 instead of 100?



Start: s_0 = d1

Goal: S_g = {d4}

Asynchronous

$Q(\text{d1},\text{m12}) = 100 + 0 = 100$

$Q(\text{d1},\text{m14})$
$= 1 + (\tfrac{1}{2}(1^3/_4) + \tfrac{1}{2}(0)) = 1^7/_8$

$V(\text{d1}) = 1^7/_8;\ \pi(\text{d1}) = \text{m14}$

$Q(\text{d2},\text{m21}) = 100 + 1^7/_8 = 101^7/_8$

$Q(\text{d2},\text{m23}) = 1 + (\tfrac{1}{2}(6) + \tfrac{1}{2}(6)) = 7$

$V(\text{d2}) = 7;\ \pi(\text{d2}) = \text{m23}$

$Q(\text{d3},\text{m32}) = 1 + 7 = 8$

$Q(\text{d3},\text{m34}) = 100 + 0 = 100$

$V(\text{d3}) = 8;\ \pi(\text{d3}) = \text{m32}$

$Q(\text{d5},\text{m52}) = 1 + 7 = 8$

$Q(\text{d5},\text{m54}) = 100 + 0 = 100$

$V(\text{d5}) = 8;\ \pi(\text{d5}) = \text{m52}$

$r = \max(1^7/_8 - 1^3/_4,\ 7 - 5,\ 8 - 6,\ 8 - 6) = 2$

Discussion

- Policy iteration computes new π in each iteration; computes V^{π} from π
  - More work per iteration than value iteration
    - Needs to solve a set of simultaneous equations
  - Usually converges in a smaller number of iterations
- Value iteration
  - Computes new V in each iteration; chooses π based on V
  - New V is a revised set of heuristic estimates
    - Not V^{π} for π or any other policy
  - Less work per iteration: doesn't need to solve a set of equations
  - Usually takes more iterations to converge

- At each iteration, both algorithms need to examine the entire state space
  - Number of iterations polynomial in $|S|$, but $|S|$ may be quite large
- Next: use search techniques to avoid searching the entire space

AO*

Requires acyclic Σ

not in book

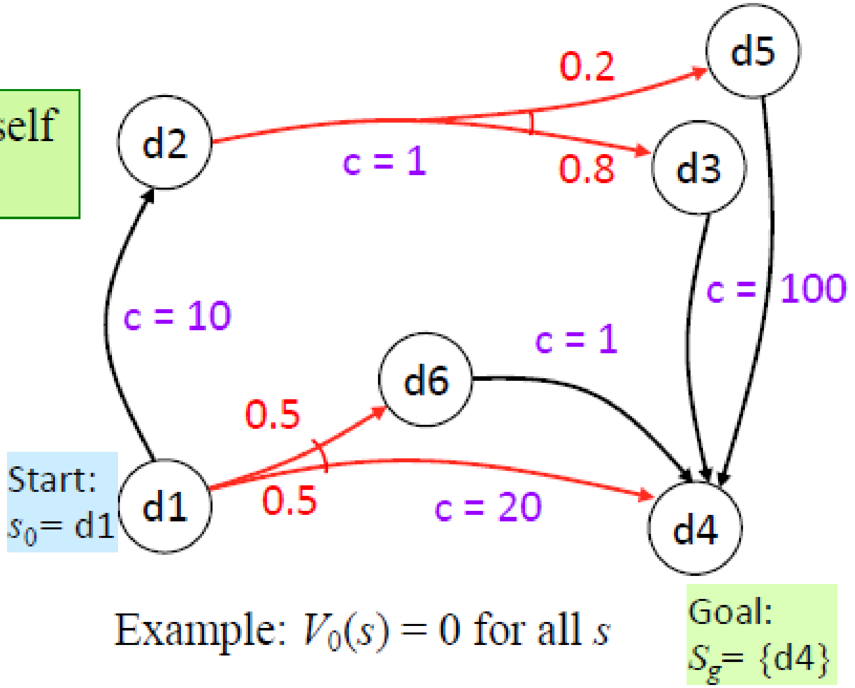
AO* (Σ, s_0, S_g, V_0)
  global $\pi \leftarrow \emptyset$; global $V(s_0) \leftarrow V(s_0)$
  global $Envelope \leftarrow \{s_0\}$ // generated states
  while leaves$(s_0,\pi) \setminus S_g \neq \emptyset$ do
    select $s \in$ leaves$(s_0,\pi) \setminus S_g$
    for all $a \in$ Applicable(s)
      for all $s' \in \gamma(s,a) \setminus Envelope$ do
        $V(s') \leftarrow V_0(s')$; add s' to $Envelope$
    AO-Update(s)
  return π

no π-descendants in Z but s itself
- ensures bottom-up updates

AO-Update(s)
  $Z \leftarrow \{s\}$ // nodes that need updating
  while $Z \neq \emptyset$ do
    select $s \in Z$ such that $\hat{\gamma}(s,\pi(s)) \cap Z = \{s\}$
    remove s from Z
    Bellman-Update(s)
    $Z \leftarrow Z \cup \{s' \in Envelope \mid s \in \gamma(s',\pi)\}$

the states "just above" s

Bellman-Update(s)
  $v_{old} \leftarrow V(s)$
  for every $a \in$ Applicable(s) do
    $Q(s,a) \leftarrow \text{cost}(s,a) + \sum_{s' \in S} \Pr(s'|s,a)\, V(s')$
  $V(s) \leftarrow \min_{a \in \text{Applicable}(s)} Q(s,a)$
  $\pi(s) \leftarrow \text{argmin}_{a \in \text{Applicable}(s)} Q(s,a)$
  return $|V(s) - v_{old}|$



Example: $V_0(s) = 0$ for all s

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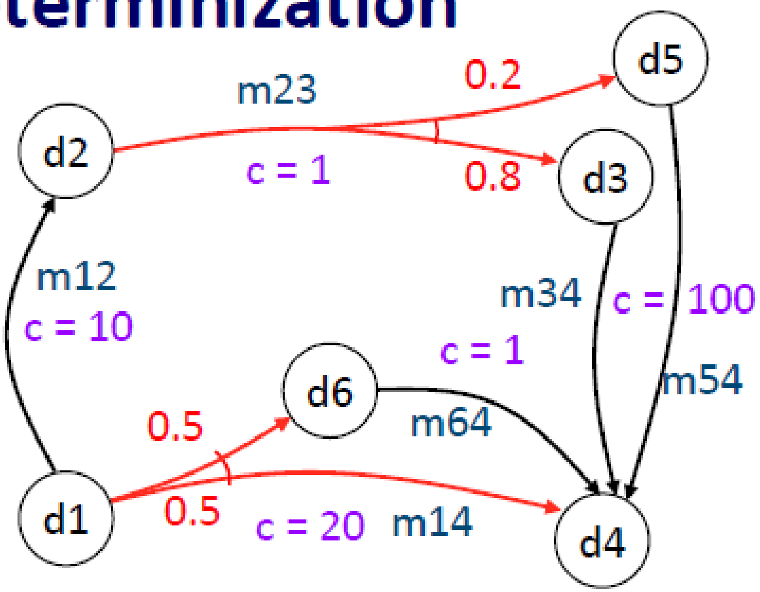
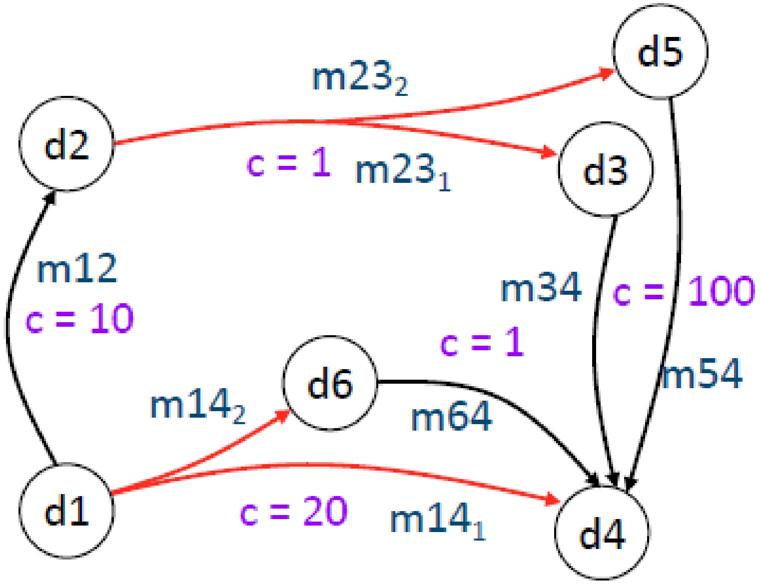
Heuristics through Determinization

- What to use for V_0?
  - One possibility: classical planner
  - Need to convert nondeterministic actions into something the classical planner can use
- *Determinize* the actions
  - Suppose $\gamma(s,a) = \{s_1, \ldots, s_n\}$
  - $\text{Det}(s,a) = \{n \text{ actions } a_1, a_2, \ldots, a_n\}$
    - $\gamma_d(s,a_i) = s_i$
    - $\text{cost}_d(s,a_i) = \text{cost}(s,a)$
- Classical domain $\Sigma_d = (S, A_d, \gamma_d, \text{cost}_d)$
  - S = same as in Σ
  - $A_d = \cup_{a \in A, s \in S} \text{Det}(s,a)$
  - γ_d and cost_d as above





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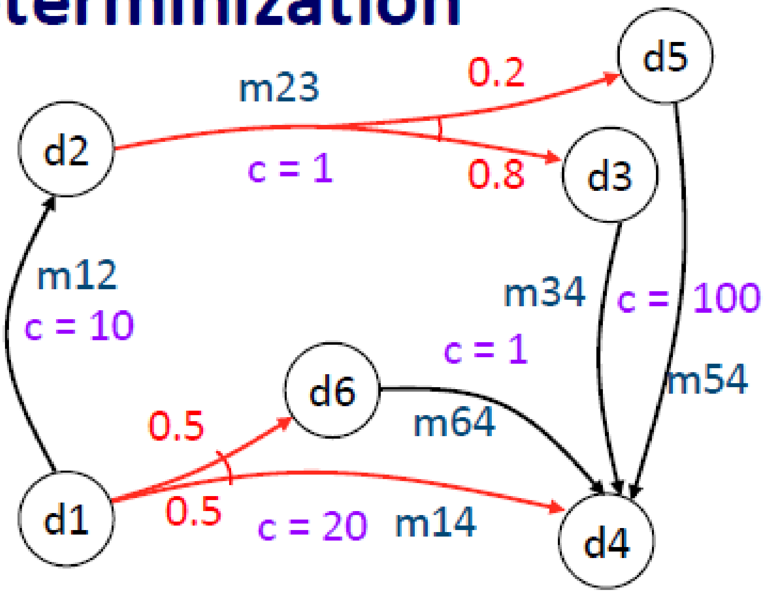
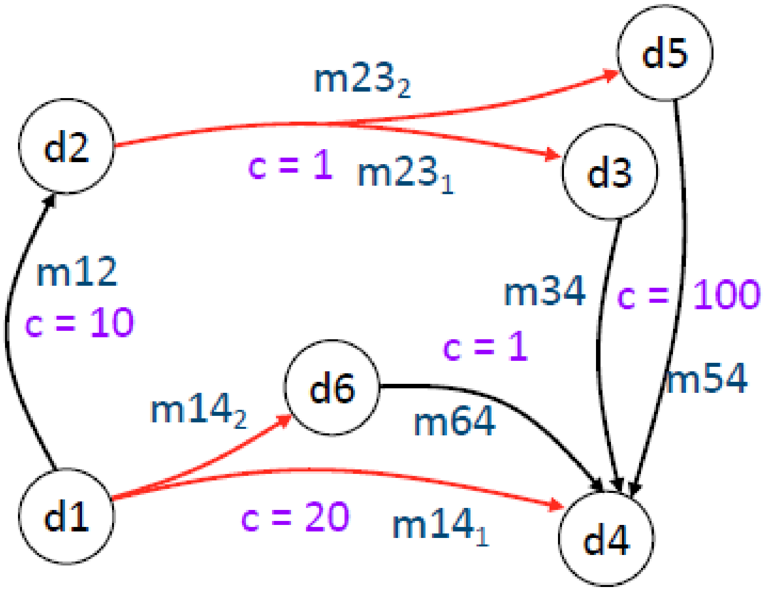
Heuristics through Determinization

- Suppose we want $V_0(s)$
- Call classical planner on (Σ_d, s, S_g)
  - ➢ Get plan $p = \langle a_1, a_2, \ldots, a_n \rangle$
  - ➢ Goes through states $\langle s, s_1, \ldots, s_n \rangle$
    - $s_1 = \gamma(s,a_1),\ s_2 = \gamma(s_1,a_2),\ \ldots$
  - ➢ Return $V_0(s) = \text{cost}(p) = \sum_i \text{cost}(a_i)$
- If the classical planner always returns optimal plans, then V_0 is admissible
- Outline of proof:
  - ➢ Let π be a safe solution in Σ
  - ➢ Every acyclic execution of π corresponds to a solution plan p' in Σ_d
    - Must have cost $\geq V_0(s)$
    - Otherwise the classical planner would have chosen p' instead of p





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Σ may be either cyclic or acyclic

LAO*

not in book

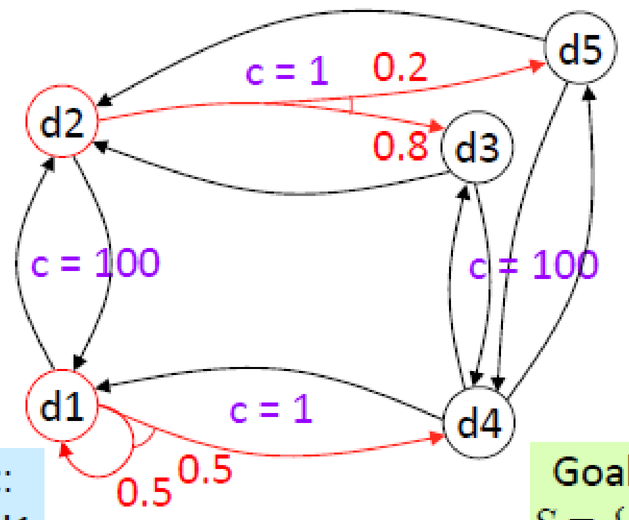
LAO*(Σ, s_0, S_g, V_0)
  global $\pi \leftarrow \emptyset$; global $V(s_0) \leftarrow V(s_0)$
  global *Envelope* $\leftarrow \{s_0\}$ // generated states
  loop
    if leaves(s_0,π) $\subseteq S_g$ then return π
    select $s \in$ leaves(s_0,π) \ S_g
    for all $a \in$ Applicable(s)
      for all $s' \in \gamma(s,a)$ \ *Envelope* do
        $V(s') \leftarrow V_0(s')$; add s' to *Envelope*
    LAO-Update(s)
  return π

All π-ancestors of s in *Envelope*

LAO-Update(s)
  $Z \leftarrow \{s\} \cup \{s' \in Envelope \mid s \in \hat{\gamma}(s',\pi)\}$
  loop
    $r \leftarrow \max_{s \in Z}$ Bellman-Update(s)
    if *leaves*(s_0,π) changed or $r \leq \eta$ then break

Asynchronous value iteration, restricted to Z

Bellman-Update(s)
  $v_{old} \leftarrow V(s)$
  for every $a \in$ Applicable(s) do
    $Q(s,a) \leftarrow \text{cost}(s,a) + \sum_{s' \in S} \Pr(s'|s,a)\, V(s')$
  $V(s) \leftarrow \min_{a \in \text{Applicable}(s)} Q(s,a)$
  $\pi(s) \leftarrow \text{argmin}_{a \in \text{Applicable}(s)} Q(s,a)$
  return $|V(s) - v_{old}|$



Start:
s_0 = d1

Goal:
S_g = {d4}

Example: $V_0(s) = 0$ for all s

LAO* Example

1st iteration of main loop:

Expand d1: add d2 and d4 to *Envelope*

Call LAO-Update(d1)

π is empty, so Z = {d1}

Iteration 1:

Q(d1,m12) = 100 + 0 = 100

Q(d1,m14) = 1 + (½(0) + ½(0)) = 1

V(d1) = 1; π(d1) = m14; r = V(d1) – 0 = 1

Iteration 2:

Q(d1,m12) = 100 + 0 = 100

Q(d1,m14) = 1 + (½(1) + ½(0)) = 1½

V(d1) = 1½; π(d1) = m14; r = 1½ – 1 = ½

Iteration 3:

Q(d1,m12) = 100 + 0 = 100

Q(d1,m14) = 1 + (½(1½) + ½(0)) = $1^3/_4$

V(d1) = 1¾; π(d1) = m14; r = 1¾ – 1½ = ¼

$\eta = 0.2$

$V_0(s) = 0$ for all s

Iteration 4:

Q(d1,m12) = 100 + 0 = 100

Q(d1,m14) = 1 + (½(1¾) + ½(0)) = $1^7/_8$

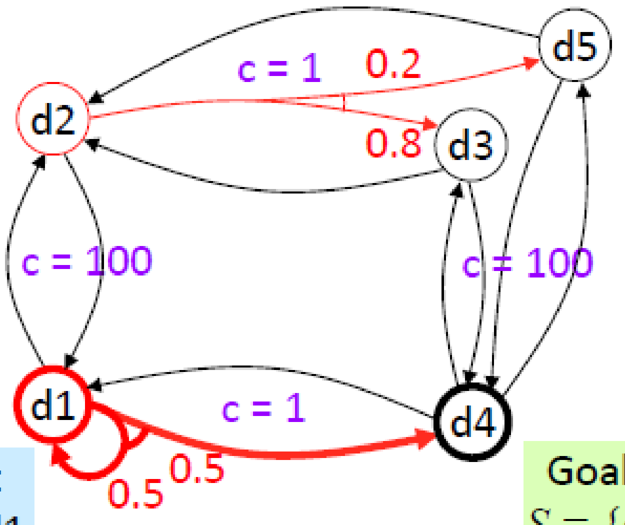
V(d1) = $1^7/_8$; π(d1) = m14; $r = {}^1/_8 \le \eta$

LAO-Update returns

2nd iteration of main loop:

leaves(π) = {d4} ⊆ S_g

return π



Start:
s_0 = d1

Goal:
S_g = {d4}

Planning and Acting

Run-Lookahead(Σ, s_0, S_g)

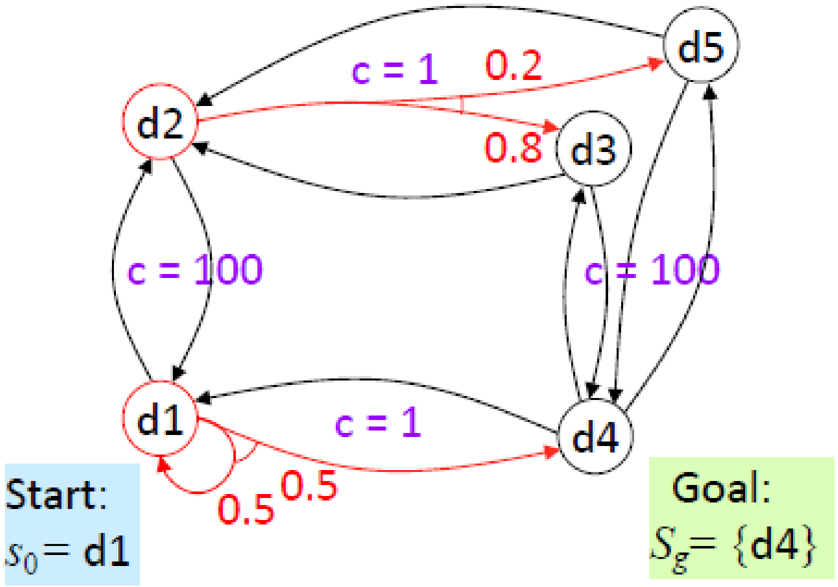
$s \leftarrow s_0$

while $s \notin S_g$ and Applicable(s) $\neq \emptyset$ do

$a \leftarrow$Lookahead(s,θ)

perform action a

$s \leftarrow$ observe resulting state



- Same as in Chapter 2, except $s = \xi$
  - Could use $s \leftarrow$ abstraction of ξ as in Chapter 2
- Could also use Run-Lazy-Lookahead or Run-Concurrent-Lookahead

- What to use for Lookahead?
  - AO*, LAO*, …
    - Modify to search part of the space
  - Classical planner running on determinized domain
  - Stochastic sampling algorithms

Planning and Acting

Run-Lookahead(Σ, s_0, S_g)

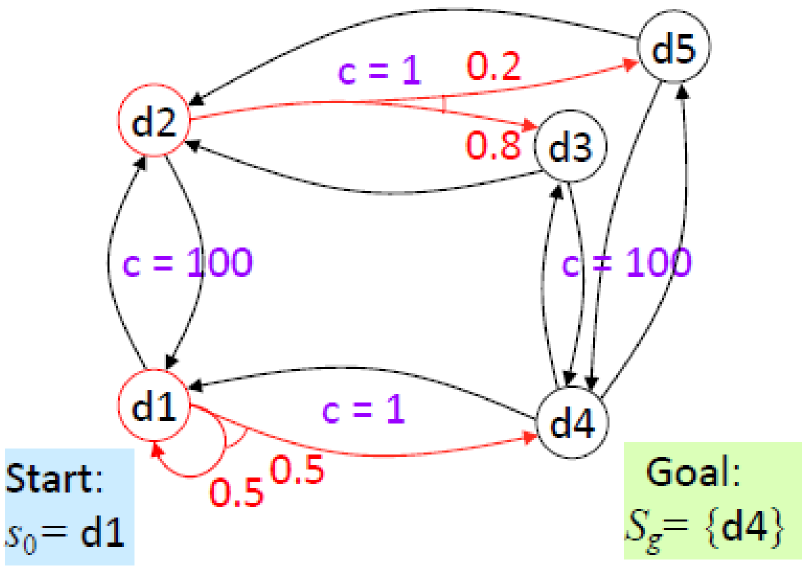
$s \leftarrow s_0$

while $s \notin S_g$ and Applicable(s) $\neq \emptyset$ do

$a \leftarrow$ Lookahead(s, θ)

perform action a

$s \leftarrow$ observe resulting state



- If Lookahead = classical planner on determinized domain

  ⇒ FS-Replan (Chapter 5)

- Problem: Forward-search may choose a plan that depends on low-probability outcome
- RFF algorithm (see book) attempts to alleviate this

FS-Replan (Σ, s, S_g)

$\pi_d \leftarrow \varnothing$

while $s \notin S_g$ and Applicable(s) $\neq \varnothing$ do

if π_d undefined for s then do

$\pi_d \leftarrow$ Forward-search (Σ_d, s, S_g)

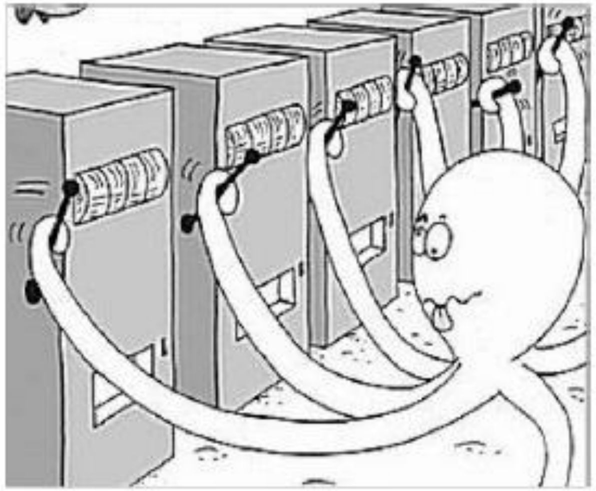
if $\pi_d =$ failure then return failure

perform action $\pi_d(s)$

$s \leftarrow$ observe resulting state

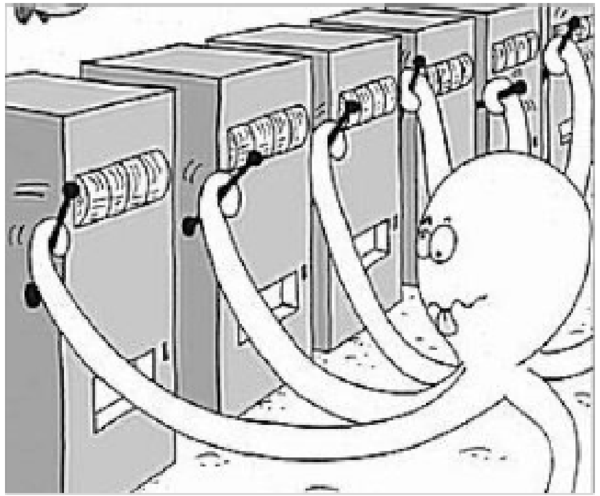
Multi-Arm Bandit Problem

- Statistical model of sequential experiments
  - Name comes from a traditional slot machine (one-armed bandit)
- Multiple actions $a_1, a_2, \ldots, a_n$
  - Each a_i provides a reward from an unknown (but stationary) probability distribution p_i
  - Objective: maximize expected utility of a sequence of actions
- Exploitation vs exploration dilemma:
  - ***Exploitation***: choose action that has given you high rewards in the past
  - ***Exploration***: choose action that you don't know much about, in hopes that it might produce a higher reward



UCB (Upper Confidence Bound) Algorithm

- Assume all rewards are between 0 and 1
  - If they aren't, normalize them
- For each action a_i , let
  - r_i = average reward you've gotten from a_i
  - t_i = number of times you've tried a_i
  - $t = \sum_i t_i$



loop

if there are one or more actions that you haven't tried

then choose an untried action a_i at random

else choose an action a_i that has the highest value of $r_i + \sqrt{2(\ln t)/t_i}$

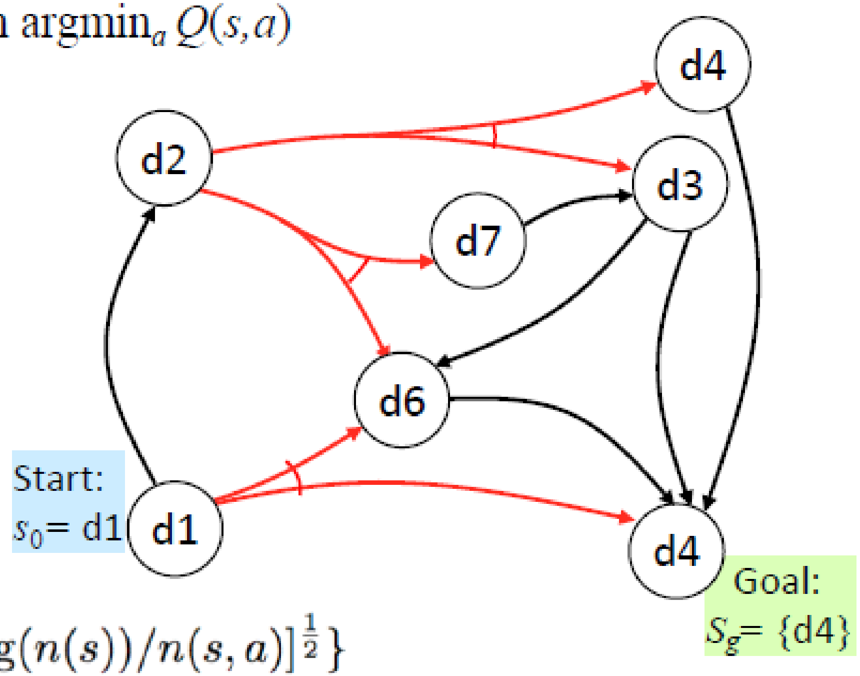
perform a_i

update r_i, t_i, t

UCT Algorithm

- Recursive UCB computation to compute $Q(s,a)$
- Anytime algorithm: call repeatedly until time runs out
  - Then choose action $\text{argmin}_a\, Q(s,a)$

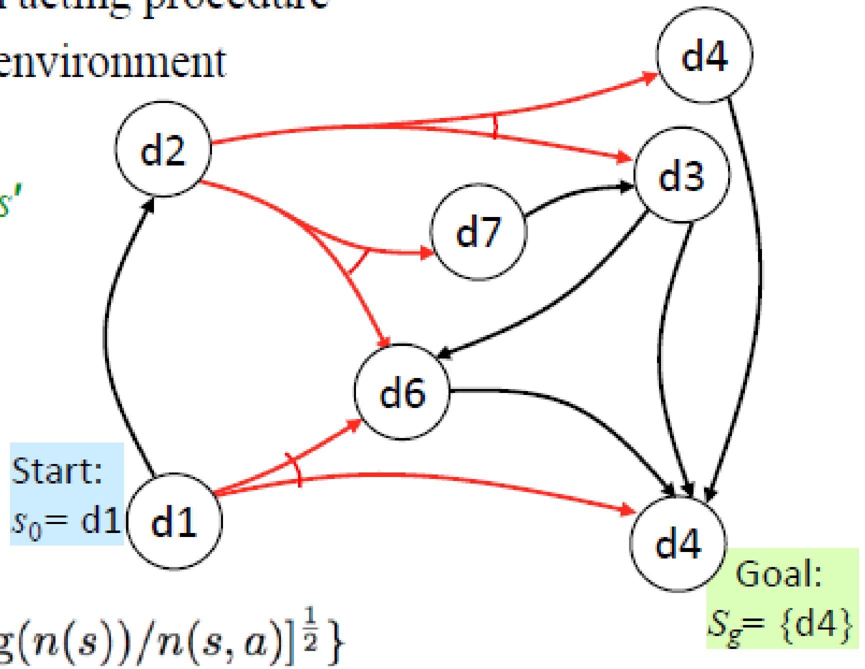
UCT(s, h)
  if $s \in S_g$ then return 0
  if $h = 0$ then return $V_0(s)$
  if $s \notin Envelope$ then do
    add s to $Envelope$
    $n(s) \leftarrow 0$
    for all $a \in \text{Applicable}(s)$ do
      $Q(s,a) \leftarrow 0;\ \ n(s,a) \leftarrow 0$
  $Untried \leftarrow \{a \in \text{Applicable}(s) \mid n(s,a) = 0\}$
  if $Untried \neq \varnothing$ then $\tilde{a} \leftarrow$ Choose$(Untried)$
  else $\tilde{a} \leftarrow \text{argmin}_{a \in \text{Applicable}(s)} \{Q(s,a) - C \times [\log(n(s))/n(s,a)]^{\frac{1}{2}}\}$
  $s' \leftarrow$ Sample$(\Sigma, s, \tilde{a})$
  $cost\text{-}rollout \leftarrow \text{cost}(s,\tilde{a}) +$ UCT$(s', h-1)$
  $Q(s,\tilde{a}) \leftarrow [n(s,\tilde{a}) \times Q(s,\tilde{a}) + cost\text{-}rollout]/(1 + n(s,\tilde{a}))$
  $n(s) \leftarrow n(s) + 1$
  $n(s,\tilde{a}) \leftarrow n(s,\tilde{a}) + 1$
  return $cost\text{-}rollout$



UCT as an Acting Procedure

- Suppose you don't know the probabilities and costs
- Suppose you can restart your actor as many times as you want
- Can modify UCT to be an acting procedure
  - Use it to explore the environment

UCT(s, h)
  if $s \in S_g$ then return 0
  if $h = 0$ then return $V_0(s)$
  if $s \notin Envelope$ then do
    add s to $Envelope$
    $n(s) \leftarrow 0$
    for all $a \in \text{Applicable}(s)$ do
      $Q(s,a) \leftarrow 0;\ \ n(s,a) \leftarrow 0$
  $Untried \leftarrow \{a \in \text{Applicable}(s) \mid n(s,a) = 0\}$
  if $Untried \neq \varnothing$ then $\tilde{a} \leftarrow$ Choose$(Untried)$
  else $\tilde{a} \leftarrow \text{argmin}_{a \in \text{Applicable}(s)} \{Q(s,a) - C \times [\log(n(s))/n(s,a)]^{\frac{1}{2}}\}$
  $s' \leftarrow$ Sample$(\Sigma, s, \tilde{a})$ ← perform a; observe s'
  $cost\text{-}rollout \leftarrow \text{cost}(s, \tilde{a}) + \text{UCT}(s', h-1)$
  $Q(s,\tilde{a}) \leftarrow [n(s,\tilde{a}) \times Q(s,\tilde{a}) + cost\text{-}rollout]/(1 + n(s,\tilde{a}))$
  $n(s) \leftarrow n(s) + 1$
  $n(s,\tilde{a}) \leftarrow n(s,\tilde{a}) + 1$
  return $cost\text{-}rollout$



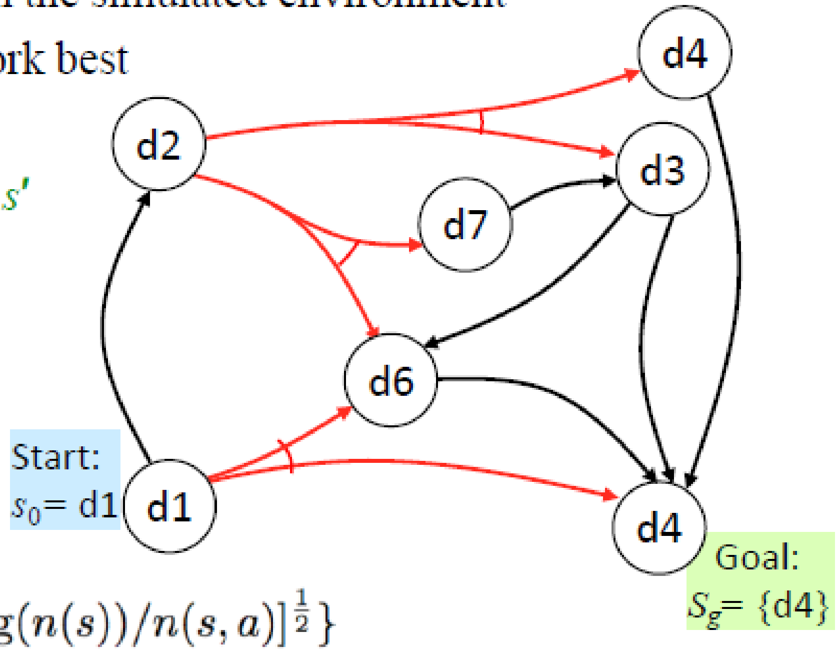
UCT as a Learning Procedure

- Suppose you don't know the probabilities and costs
  - But you have an accurate simulator for the environment
- Run UCT multiple times in the simulated environment
  - Learn what actions work best

$\mathsf{UCT}(s, h)$

  if $s \in S_g$ then return 0
  if $h = 0$ then return $V_0(s)$
  if $s \notin \mathit{Envelope}$ then do
    add s to $\mathit{Envelope}$
    $n(s) \leftarrow 0$
    for all $a \in \text{Applicable}(s)$ do
      $Q(s,a) \leftarrow 0;\ \ n(s,a) \leftarrow 0$
  $\mathit{Untried} \leftarrow \{a \in \text{Applicable}(s) \mid n(s,a) = 0\}$
  if $\mathit{Untried} \neq \varnothing$ then $\tilde{a} \leftarrow \mathsf{Choose}(\mathit{Untried})$
  else $\tilde{a} \leftarrow \text{argmin}_{a \in \text{Applicable}(s)} \{Q(s,a) - C \times [\log(n(s))/n(s,a)]^{\frac{1}{2}}\}$
  $s' \leftarrow \mathsf{Sample}(\Sigma, s, \tilde{a})$
  $\mathit{cost\text{-}rollout} \leftarrow \text{cost}(s,\tilde{a}) + \mathsf{UCT}(s', h-1)$
  $Q(s,\tilde{a}) \leftarrow [n(s,\tilde{a}) \times Q(s,\tilde{a}) + \mathit{cost\text{-}rollout}]/(1 + n(s,\tilde{a}))$
  $n(s) \leftarrow n(s) + 1$
  $n(s,\tilde{a}) \leftarrow n(s,\tilde{a}) + 1$
  return $\mathit{cost\text{-}rollout}$

simulate a; observe s'



UCT in Two-Player Games

- Generate Monte Carlo rollouts using a modified version of UCT
- Main differences:
  - Instead of choosing actions that minimize accumulated cost, choose actions that maximize payoff at the end of the game
  - UCT for player 1 recursively calls UCT for player 2
    - Choose opponent's action
  - UCT for player 2 recursively calls UCT for player 1

- This produced the first computer programs to play go well
  - ≈ 2008–2012
- Monte Carlo rollout techniques similar to UCT were used to train AlphaGo

Summary

- SSPs
- solutions, closed solutions, histories
- unsafe solutions, acyclic safe solutions, cyclic safe solutions
- expected cost, planning as optimization
- policy iteration
- value iteration (synchronous, asynchronous)
  - Bellman-update
- AO*, LAO*
- Planning and Acting
  - Run-Lookahead
  - RFF
- UCB, UCT