Classifiers: Naïve Bayes, evaluation

Tomáš Svoboda and Matěj Hoffmann
thanks to Daniel Novák and Filip Železný, Ondřej Drbohlav

Vision for Robots and Autonomous Systems, Center for Machine Perception
Department of Cybernetics
Faculty of Electrical Engineering, Czech Technical University in Prague

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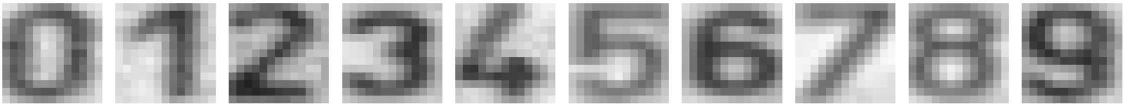
Notes

Example: Digit recognition/classification



- Input: 8-bit image 13×13, pixel intensities $0 - 255$. (0 means black, 255 means white)
- Output: Digit $0 - 9$. Decision about the class, classification.
- Features: Pixel intensities ...

Decision/classification problem : What cipher is in the (query) image?

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Digit recognition is a very classical example of classification problem. It has been used for decades, and it is used till today, see e.g. MNIST demo at PyTorch

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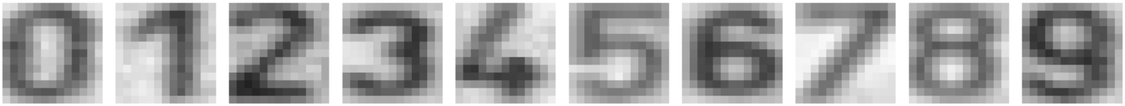

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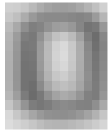
Decision/classification problem : What cipher is in the (query) image? 

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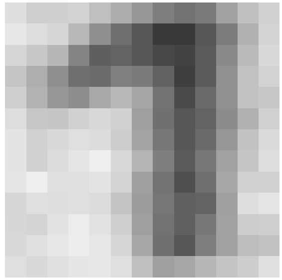
Classification as a special case of statistical decision theory



- Attribute vector $\vec{x} = [x_1, x_2, \ldots]^\top$: pixels 1, 2, ...
- **State set $\mathcal{S}$ = decision set $\mathcal{D} = \{0, 1, \ldots 9\}$.**
- State = actual class, Decision = recognized class
- Loss function: $l(s,d) = \begin{cases} 0, & d = s \\ 1, & d \neq s \end{cases}$

Optimal decision strategy:

$$\delta^*(\vec{x}) = \arg\min_d \sum_s \underbrace{l(s,d)}_{0 \text{ if } d=s} P(s|\vec{x}) = \arg\min_d \sum_{s \neq d} P(s|\vec{x})$$

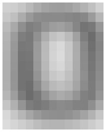
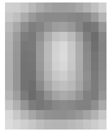
Obviously $\sum_s P(s|\vec{x}) = 1$, then: $P(d|\vec{x}) + \sum_{s \neq d} P(s|\vec{x}) = 1$
Inserting into above:

$$\delta^*(\vec{x}) = \arg\min_d \Big(1 - P(d|\vec{x})\Big) = \arg\max_d P(d|\vec{x})$$

Notes

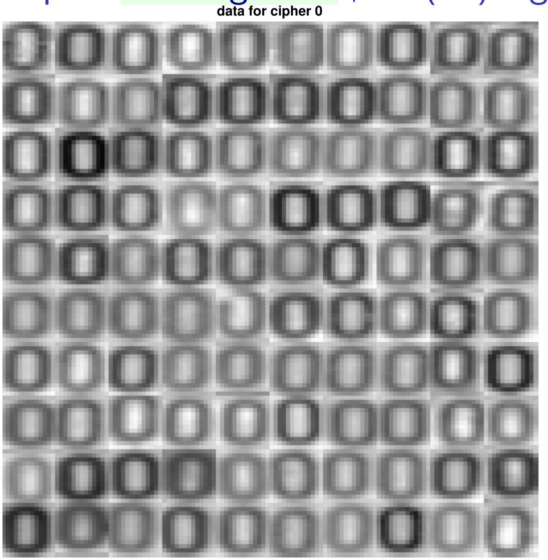
We are using different word – *classification* instead of *decision* but the reasoning and methods can be well applied in both. In classification problem we usually treat all mistakes – wrong classificaions – equally painful, contrary to decision problem – remember "What to cook for dinner" problem?

Optimal (Bayes) Classification

$\delta^*($  $) = \arg\max_d P(d|$  $)$

Notes

Machine Learnine: Prepare training data , let (an) algorithm learn itself



Training samples: $(\vec{x}_i, s = 0)$

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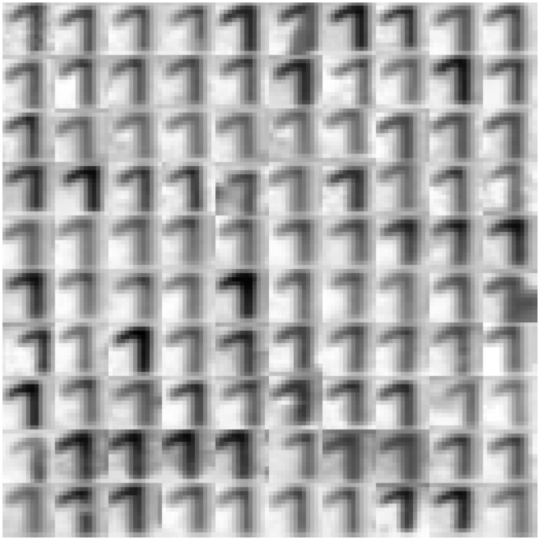

Training samples: $(\vec{x}_i, s = 1)$

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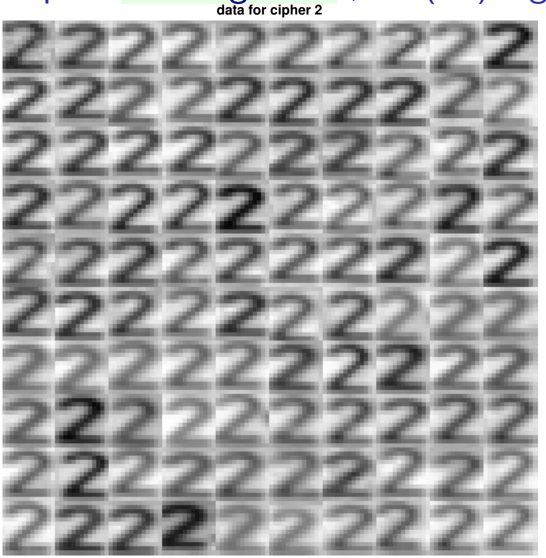


Training samples: $(\vec{x}_i, s = 2)$

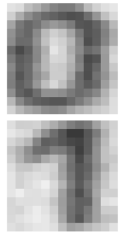
Notes

Bayes classification in practice; $P(s|\vec{x}) = ?$

- Usually, we are not given $P(s|\vec{x})$
- It has to be estimated from already classified examples – training data
- For discrete $\vec{x}$, training examples $(\vec{x}_1, s_1), (\vec{x}_2, s_2), \ldots (\vec{x}_l, s_l)$
  - every $(\vec{x}_i, s)$ is drawn independently from $P(\vec{x}, s)$, i.e. sample i does not depend on $1, \cdots, i-1$
  - so-called i.i.d (independent, identically distributed) multiset
- Without knowing anything about the distribution, a non-parametric estimate:

$$P(s|\vec{x}) = \frac{P(\vec{x}, s)}{P(\vec{x})} \approx \frac{\#\text{ examples where } \vec{x}_i = \vec{x} \textbf{ and } s_i = s}{\#\text{ examples where } \vec{x}_i = \vec{x}}$$

- Hard in practice:
  - To reliably estimate $P(s|\vec{x})$, the number of examples grows exponentially with the number of elements of $\vec{x}$.
    - e.g. with the number of pixels in images
    - curse of dimensionality
    - denominator often 0

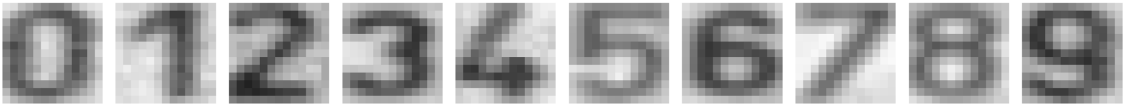


Notes

Why hard? Way too many various $\vec{x}$.
What is the difference between set and multiset?

Reminder about math notation. In literature, vectors are mostly denoted by bold lower case **x**. In lectures, we use $\vec{x}$ to match notation used on blackboard. It is difficult to write bold with a chalk.

How many images?



8-bit image 13×13, pixel intensities $0 - 255$. (0 means black, 255 means white)

A: 169^{256}

B: 256^{169}

C: 13^{13}

D: 169×256

E: different quantity

Notes

Naïve Bayes classification

- For efficient classification we must thus rely on additional assumptions.
- In the exceptional case of statistical independence between components of $\vec{x}$ for each class s it holds

$$P(\vec{x}|s) = P(x[1]|s) \cdot P(x[2]|s) \cdot \ldots$$

- Use simple Bayes law and maximize:

$$P(s|\vec{x}) = \frac{P(\vec{x}|s)P(s)}{P(\vec{x})} = \frac{P(s)}{P(\vec{x})} P(x[1]|s) \cdot P(x[2]|s) \cdot \ldots =$$

- No combinatorial curse in estimating $P(s)$ and $P(x[i]|s)$ separately for each i and s.
- No need to estimate $P(\vec{x})$. (Why?)
- $P(s)$ may be provided apriori.
- naïve = when used despite statistical dependence

Notes

Why naïve at all? Consider $N-$dimensional feature space and $8-bit$ values. Instead of considering 8^N combinations (joint prob. distribution), we can consider only $N \times 8$—treating every feature separately.

Think about statistical independence. Example1: person's weight and height. Are they independent? Example2: pixel values in images.

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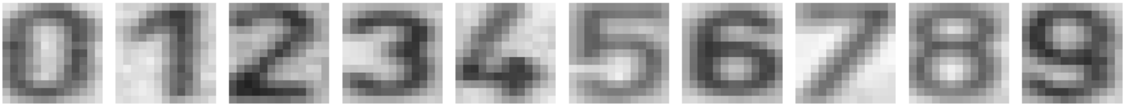

- **Input:** 8-bit image 13×13, pixel intensities $0 - 255$. (0 means black, 255 means white)
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- **Features:** Pixel intensities ...

Collect data , ...

- $P(\vec{x})$. What is the dimension of $\vec{x}$? How many possible images?
- Learn $P(\vec{x}|s)$ per each class (digit).
- Classify $s^* = \text{argmax}_s P(s|\vec{x})$.

Notes

We can create many more features than just pixel intensities. But first things first.
We are assuming all errors are equally important - minimizing the number of wrong decisions.
Dimension of $\vec{x}$ is $13 \times 13 = 169$. There are 256^{169} possible images. (we already know)

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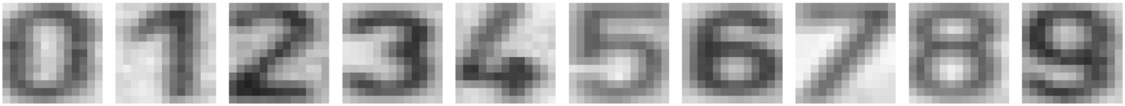

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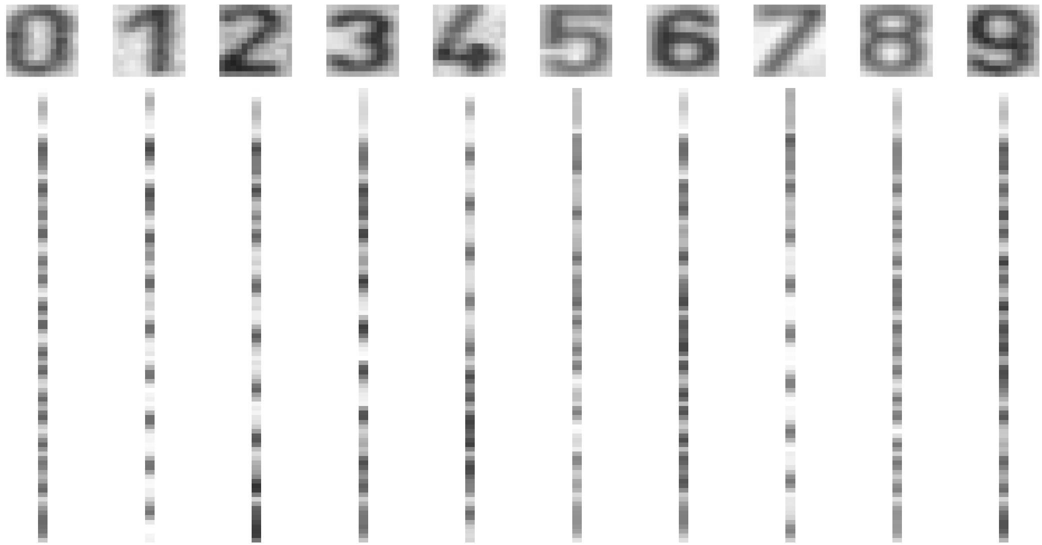
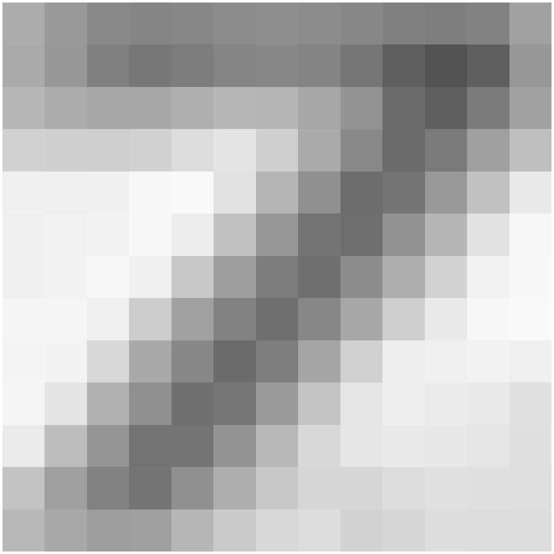
From images to $\vec{x}$



Notes

Conditional probabilities, likelihoods



- Apriori digit probabilities $P(s_k)$
- Likelihoods for pixels. $P(x_{r,c} = l_i | s_k)$

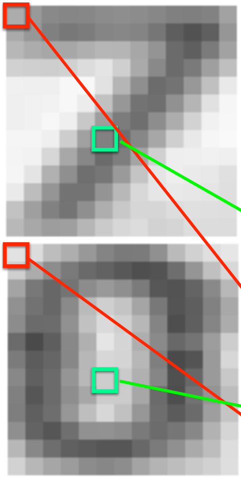
Notes

A lexical note, especially for Czech speakers. *probability* as well as *likelihood* can be translated as *pravděpodobnost*. I suggest the following mental model than can work for our purposes.

- Probability is related to the future events (unknown outcome). E.g. what is the probability of selecting blue box? What is the probability that a random ZIP Code number begins with 7?
- Likelihood refers to past events (known outcome). In my data, how many images of 7 have dark pixel in top right corner? We can think about relative frequency (relativní četnost).

Conditional likelihoods

s_k	$P(s_k)$
1	0.1
2	0.1
3	0.1
4	0.1
5	0.1
6	0.1
7	0.1
8	0.1
9	0.1
0	0.1



$P(x_{0,0} = 25|s_k)$

s_k	$P(x_{0,0} = 25|s_k)$
1	0.0009
2	0.05
3	0.00006
4	0.0005
5	0.07
6	0.0007
7	0.1
8	0.0006
9	0
0	0

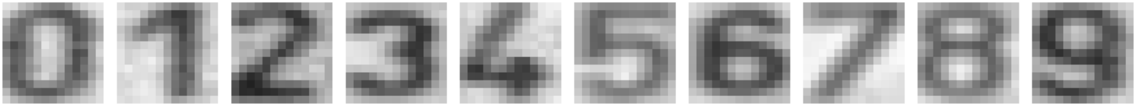
$P(x_{7,5} = 25|s_k)$

s_k	$P(x_{7,5} = 25|s_k)$
1	0.0009
2	0.05
3	0.00006
4	0.0005
5	0.004
6	0.07
7	0.1
8	0.006
9	0.08
0	0.005

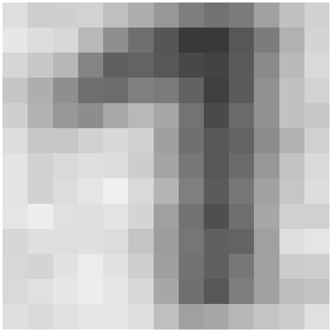
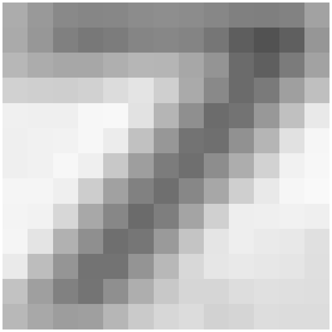
Notes

For each pixel (position) and possible instensity (image/pixel value) we create such a table.

Unseen events



Images 13×13, intensities $0 - 255$, 100 exemplars per each class.

$$\begin{aligned} \vdots &= \vdots \\ P(x_{0,0} = 100 \mid s = 7) &= 0.05 \\ P(x_{0,0} = 101 \mid s = 7) &= 0 \\ P(x_{0,0} = 102 \mid s = 7) &= 0.06 \\ \vdots &= \vdots \end{aligned}$$

A new (not in training) query image with $x_{0,0} = 101$. How would you classify?

Notes

Think about the problem of classifying numerals. Some $P(x_{r,c} = I \mid s) = 0$. What about an example:

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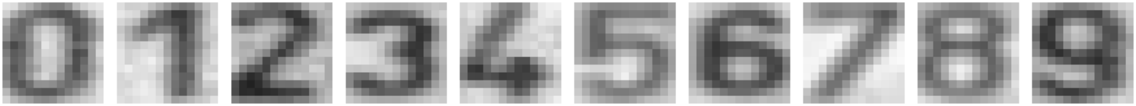

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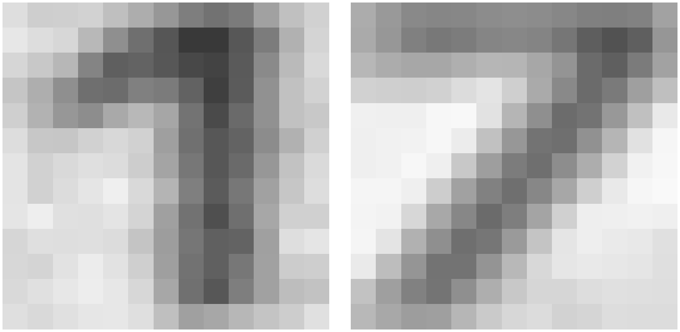

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Unseen event, how to decide?

A new (not in training) query image with $x_{0,0} = 101$. How would you classify?

$$P(x_{0,0} = 101 \mid s_j) = 0, \text{ for all classes}$$

Notes

Laplace smoothing ("additive smoothing")

Think about a particular pixel with intensity x

$$P(x) = \frac{\text{count}(x)}{\text{total samples}}$$

Problem: $\text{count}(x) = 0$

Pretend you see the (any) sample one more time.

$$P_{\text{LAP}}(x) = \frac{c(x) + 1}{\sum_x [c(x) + 1]}$$

$$P_{\text{LAP}}(x) = \frac{c(x) + 1}{N + |X|}$$

where N is the number of (total) observations; $|X|$ is the number of possible values X can take (cardinality).

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Laplace smoothing - as a hyperparameter k

Pretend you see every sample k extra times:

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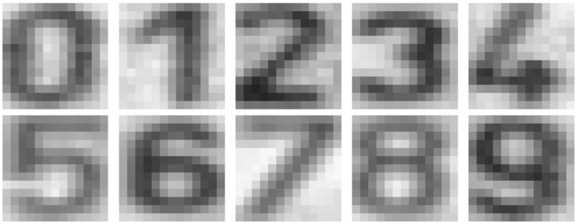
$$P_{\text{LAP}}(x) = \frac{c(x) + k}{N + k|X|}$$

For conditional, smooth each condition independently

$$P_{\text{LAP}}(x|s) = \frac{c(x,s) + k}{c(s) + k|X|}$$



What is $|X|$ equal to?

A: 10

B: 2

C: 256

D: None of the above

Notes

Hyperparameter would be tuned along with your classifier
For $k = 100$ and blue and red, you would get:

- $P_{LAP}(red) = (2 + 100)/(3 + 100 * 2) = 102/203$
- $P_{LAP}(blue) = (1 + 100)/(3 + 100 * 2) = 101/203$

In this case, smoothing ("prior") would dominate over the observations - shifting estimate from empirical to uniform.

In the digit recognition from pixels example: 256 intensity values; $13 \times 13 = 169$ pixels: Applying Laplace smoothing with $k = 1$ to $P(x)$ (prior probability of a particular pixel will take an intensity value i): $P(x_{r,c} = i) = (c(x) + 1)/(N + 256)$

Conditional: relevant for the Naïve Bayes case.

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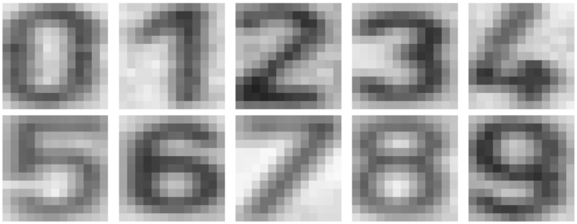

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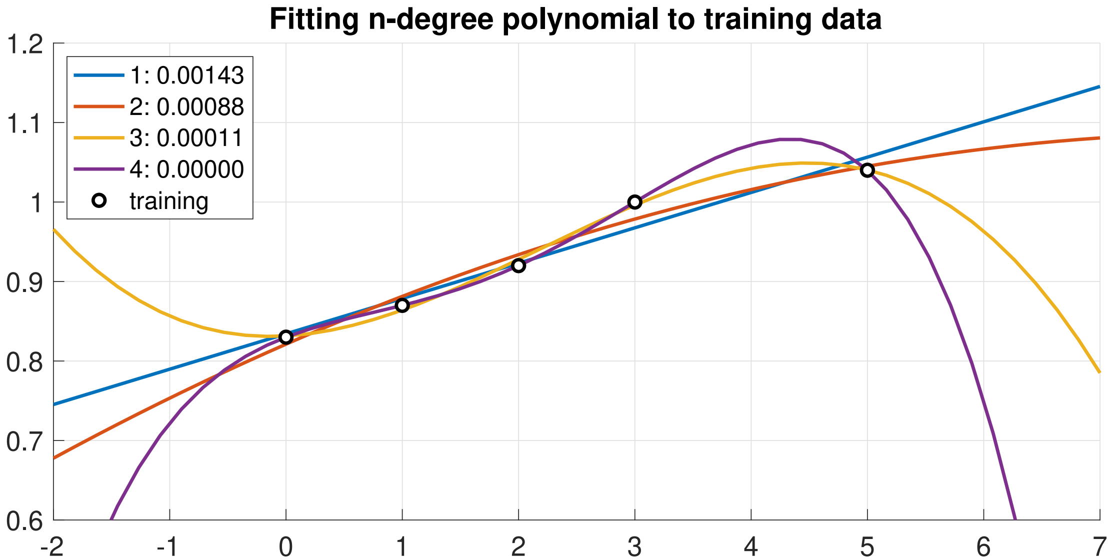
What is the right degree of polynomial (hyperparameter of a regressor)



Notes

See the `tuning_hyper_parameter.m` demo. The small values depict sum of square errors on *training data*.

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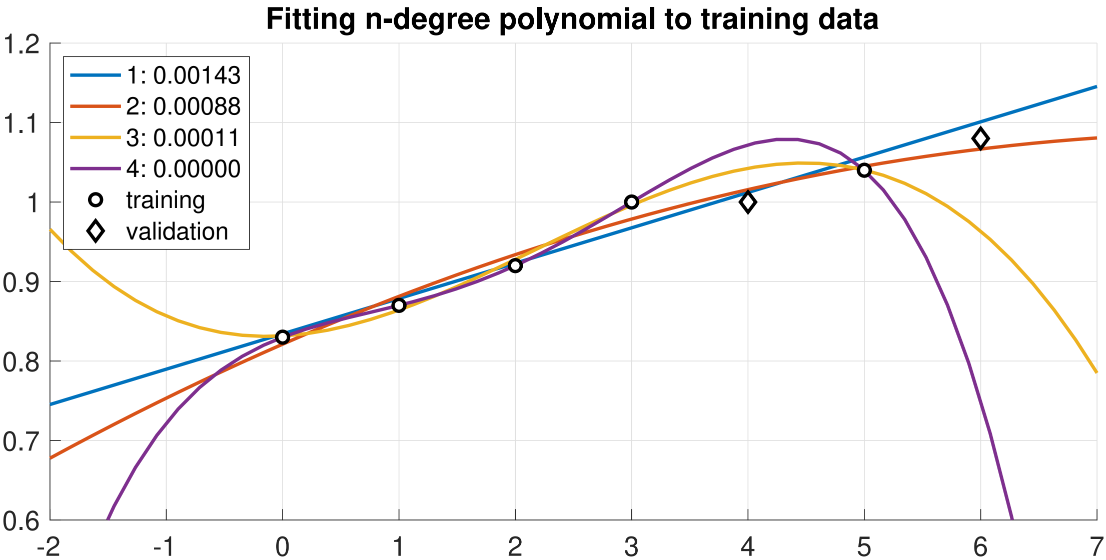

Notes

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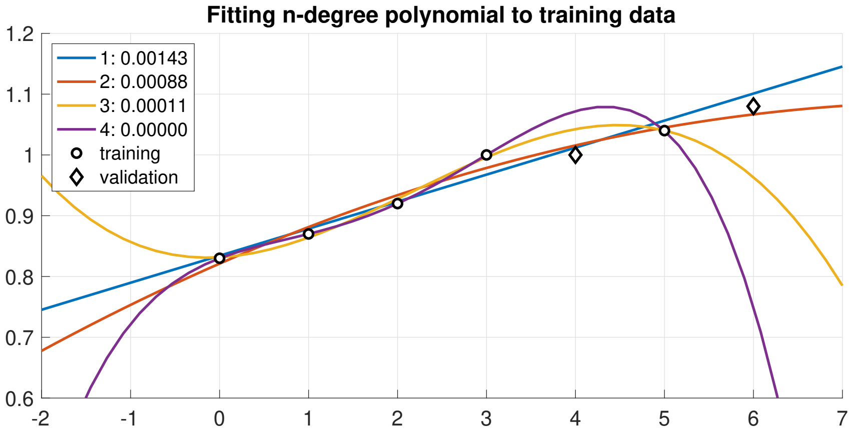
Generalization and overfiting

- Data: training, validating, testing . Wanted classifier performs well on what data?
- Overfitting: too close to training, poor on testing.



Notes

Training and testing

Data labeled instances.

- Training set
- Held-out (validation) set
- Testing set.

Features : Attribute-value pairs.

Learning cycle:

- Learn parameters (e.g. probabilities) on training set.
- Tune hyperparameters on held-out (validation) set.
- Evaluate performance on testing set.

Notes

Training set - biggest part.

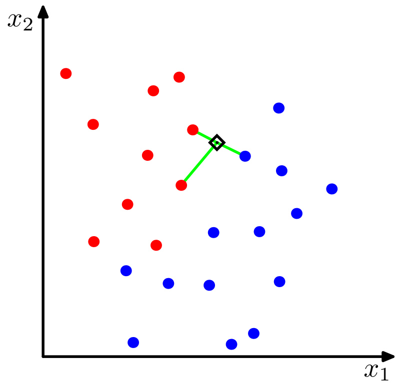
$K-$ Nearest Neighbor and Bayes $j^* = \text{argmax}_j \, P(s_j|\vec{x})$

Assume data:

- N samples $\vec{x}$ in total.
- N_j samples in s_j class. Hence, $\sum_j N_j = N$.

We want classify to $\vec{x}$. We draw a circle (hypher-sphere) centered at $\vec{x}$ containing K points irrespective of class. V is the volume of this sphere. $P(s_j|\vec{x}) =?$

$$P(s_j|\vec{x}) = \frac{P(\vec{x}|s_j)P(s_j)}{P(\vec{x})}$$



(a)

Notes

$k - NN$ for non-parametric density estimation

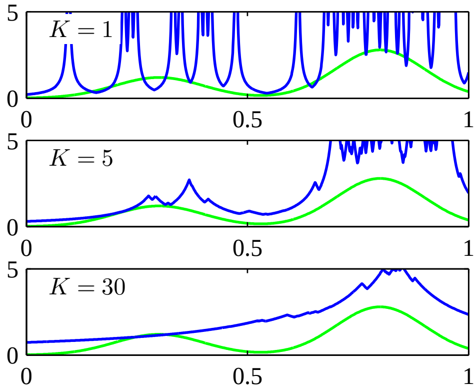
$$P(\vec{x}) = \frac{K}{NV}$$

$$V = V_d R_k^d(\vec{x})$$

$R_k(\vec{x})$ - distance from $\vec{x}$ to its $k-$th nearest neighbour point (radius)

$$V_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$$

volume od unit $d-$dimensional sphere,
Γ denotes gamma function. $V_1 = 2, V_2 = \pi, V_3 = \frac{4}{3}\pi$



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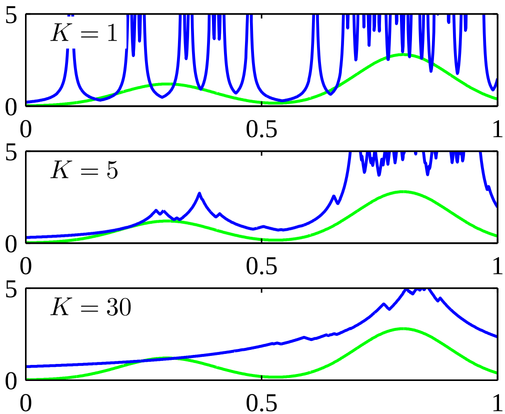


More details, including a computational example, in [2].
A $K-$NN belongs to non-parametric methods for density estimation, see section 2.5 from [1]. (Figure from [1])
Try yourself, https://scikit-learn.org/stable/modules/density.html#kernel-density

How to evaluate a classifier? Confusion table

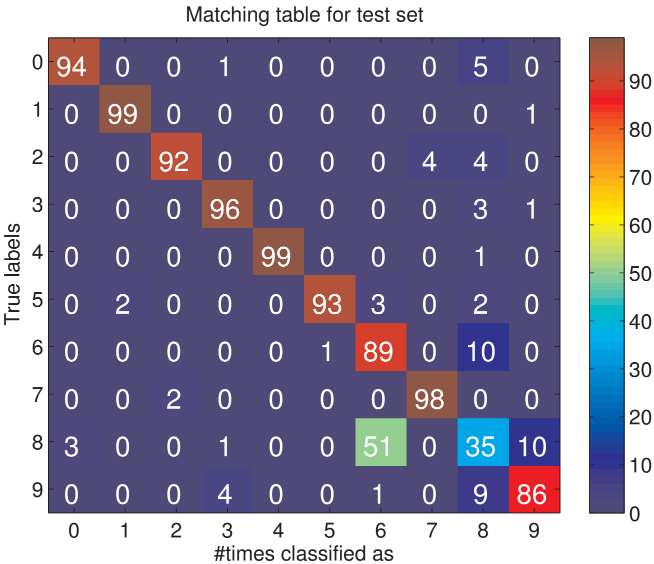


Figure from [6]

Notes

A result for a one particular classifer and its setting (parameters), one particular testing set.

Precision and Recall, and ...

Consider digit detection (is there a digit?) or SPAM/HAM classification.

Recall :

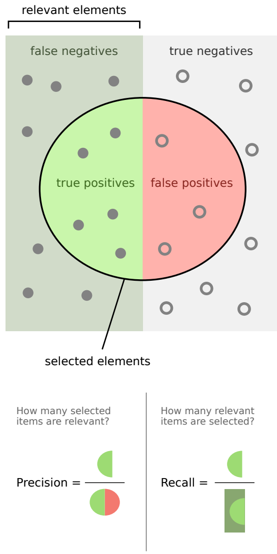
- How many relevant items are selected?
- Are we missing some items?
- Also called: True positive rate (TPR), sensitivity, hit rate ...

Precision

- How many selected items are relevant?
- Also called: Positive predictive value

False positive rate (FPR)

- Probability of false alarm



Notes

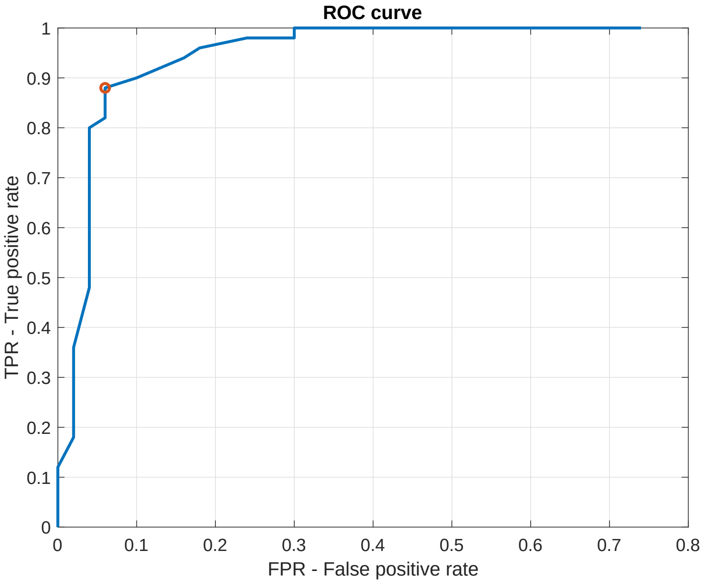
$$\text{TPR} = \frac{\text{TP}}{P} = \frac{\text{TP}}{\text{TP} + \text{FN}}$$

$$\text{Precision} = \frac{\text{TP}}{\text{TP} + \text{FP}}$$

$$\text{FPR} = \frac{\text{FP}}{N} = \frac{\text{FP}}{\text{FP} + \text{TN}}$$

Think about TPR vs FPR graph, what is the best classifier?

ROC – Receiver operating characteristics curve



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$$\text{FPR} = \frac{\text{FP}}{N} = \frac{\text{FP}}{\text{FP} + \text{TN}}$$

Notes

- How do you slide along the curve?
- What is the meaning of the diagonal?
- What would be the shape of the curve for the ideal/worst classifier?
- How would you compare various curve and select the best classifier?
- Think/read about other ways to evaluate/visualise classification results.

Product of many small numbers . . .

$$P(s|\vec{x}) = \frac{P(\vec{x}|s)P(s)}{P(\vec{x})} = \frac{P(s)}{P(\vec{x})} P(x[1]|s) \cdot P(x[2]|s) \cdot \ldots$$

$P(\vec{x})$ not needed, ...

$$\log(P(x[1]|s)P(x[2]|s)\cdots) = \log(P(x[1]|s)) + \log(P(x[2]|s)) + \cdots$$

Notes

just try

- `prod(rand(1,100))` and `prod(rand(1,10000))` in Matlab.
- `prod(rand(1,100)) == 0` and `prod(rand(1,10000)) == 0` in Matlab.

or in python console:

- `>>> import numpy as np`
- `>>> np.prod(np.random.rand(100))==0`
- `>>> np.prod(np.random.rand(1000))==0`
-
```
>>> a = np.random.rand(1000)
>>> b = np.random.rand(1000)
>>> np.prod(a)>np.prod(b)
False
>>> np.prod(a)<np.prod(b)
False
>>> np.sum(np.log(a))>np.sum(np.log(b))
True
```

Hitting the limit of number representation.
What is the way out?
$P(\vec{x})$ not needed – does not depend on the class.
Laws of logarithms...

Product of many small numbers ...

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Hitting the limit of number representation.
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$P(\vec{x})$ not needed – does not depend on the class.
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  ```

References I

Further reading: Chapter 13 and 14 of [5]. Books [1] and [3] are classical textbooks in the field of pattern recognition and machine learning. This lecture has been also inspired by the 21st lecture of CS 188 at http://ai.berkeley.edu (e.g., Laplace smoothing). Many Matlab figures created with the help of [4].

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Notes

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Notes