Electromagnetic Field Theory 2
(fundamental relations and definitions)

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Maxwell('s)-Lorentz('s) Equations

$$\nabla \times \boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{J}(\boldsymbol{r},t) + \frac{\partial \boldsymbol{D}(\boldsymbol{r},t)}{\partial t}$$

$$\nabla \times \boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t}$$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r},t) = 0$$

$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r},t) = \rho(\boldsymbol{r},t)$$

Equations of motion for fields

Equation of motion for particles

$$\boldsymbol{f}(\boldsymbol{r},t) = \rho(\boldsymbol{r},t)\boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{J}(\boldsymbol{r},t) \times \boldsymbol{B}(\boldsymbol{r},t)$$

Interaction with materials

$$\boldsymbol{D}(\boldsymbol{r},t) = \varepsilon_0 \boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{P}(\boldsymbol{r},t)$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \mu_0 \left(\boldsymbol{H}(\boldsymbol{r},t) + \boldsymbol{M}(\boldsymbol{r},t)\right)$$

Absolute majority of things happening around us is described by these equations

Boundary Conditions

$$\boldsymbol{n}(\boldsymbol{r})\times\left[\boldsymbol{E}_1(\boldsymbol{r},t)-\boldsymbol{E}_2(\boldsymbol{r},t)\right]=0$$

$$\boldsymbol{n}(\boldsymbol{r})\times\left[\boldsymbol{H}_1(\boldsymbol{r},t)-\boldsymbol{H}_2(\boldsymbol{r},t)\right]=\boldsymbol{K}(\boldsymbol{r},t)$$

$$\boldsymbol{n}(\boldsymbol{r})\cdot\left[\boldsymbol{B}_1(\boldsymbol{r},t)-\boldsymbol{B}_2(\boldsymbol{r},t)\right]=0$$

$$\boldsymbol{n}(\boldsymbol{r})\cdot\left[\boldsymbol{D}_1(\boldsymbol{r},t)-\boldsymbol{D}_2(\boldsymbol{r},t)\right]=\sigma(\boldsymbol{r},t)$$

Normal pointing to region (1)

Electromagnetic Potentials

Lorentz(‘s) calibration

$$\nabla \cdot \boldsymbol{A}(\boldsymbol{r},t) = -\sigma\mu\varphi(\boldsymbol{r},t) - \varepsilon\mu\frac{\partial\varphi(\boldsymbol{r},t)}{\partial t}$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \nabla \times \boldsymbol{A}(\boldsymbol{r},t)$$

$$\boldsymbol{E}(\boldsymbol{r},t) = -\nabla\varphi(\boldsymbol{r},t) - \frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t}$$

Lorentz(‘s) calibration depends on particular form of material relations.

Wave Equation

Material parameters are assumed independent of frequency

$$\Delta \boldsymbol{A}\left(\boldsymbol{r},t\right) - \sigma\mu \frac{\partial \boldsymbol{A}\left(\boldsymbol{r},t\right)}{\partial t} - \varepsilon\mu \frac{\partial^2 \boldsymbol{A}\left(\boldsymbol{r},t\right)}{\partial t^2} = -\mu \boldsymbol{J}_{\text{source}}\left(\boldsymbol{r},t\right)$$

Material parameters are assumed independent of coordinates

Poynting('s)-Umov('s) Theorem

Power passing the bounding envelope

Energy storage

$$-\int_V \boldsymbol{E}\cdot\boldsymbol{J}_{\text{source}}\,\mathrm{d}V = \oint_S \left(\boldsymbol{E}\times\boldsymbol{H}\right)\cdot\mathrm{d}\boldsymbol{S} + \int_V \sigma\left|\boldsymbol{E}\right|^2\mathrm{d}V + \frac{1}{2}\frac{\partial}{\partial t}\int_V\left(\varepsilon\left|\boldsymbol{E}\right|^2+\mu\left|\boldsymbol{H}\right|^2\right)\mathrm{d}V$$

Power supplied by sources

Heat losses

Energy balance in an electromagnetic system

Frequency Domain

$$\boldsymbol{F}\left(\boldsymbol{r},t\right)\in\mathbb{R} \qquad \hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right)\in\mathbb{C}$$

$$\boldsymbol{F}\left(\boldsymbol{r},t\right)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right)\mathrm{e}^{\mathrm{j}\omega t}d\omega \quad \Longleftrightarrow \quad \hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right)=\int\limits_{-\infty}^{\infty}\boldsymbol{F}\left(\boldsymbol{r},t\right)\mathrm{e}^{-\mathrm{j}\omega t}dt$$

Time derivatives reduce to algebraic multiplication

$$\frac{\partial\boldsymbol{F}\left(\boldsymbol{r},t\right)}{\partial t}\leftrightarrow\mathrm{j}\omega\hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right)$$

Spatial derivatives are untouched

$$\frac{\partial\boldsymbol{F}\left(\boldsymbol{r},t\right)}{\partial r_{\xi}}\leftrightarrow\frac{\partial\hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right)}{\partial r_{\xi}}$$

Frequency domain helps us to remove explicit time derivatives

Phasors

$$\hat{\boldsymbol{F}}\left(\boldsymbol{r},-\omega\right)=\hat{\boldsymbol{F}}^{*}\left(\boldsymbol{r},\omega\right) \quad \Rightarrow \quad \boldsymbol{F}\left(\boldsymbol{r},t\right)=\frac{1}{\pi}\int\limits_{0}^{\infty}\mathrm{Re}\left[\hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right)\mathrm{e}^{\mathrm{j}\omega t}\right]d\omega$$

Reduced frequency domain representation

Maxwell('s) Equations – Frequency Domain

$$\nabla \times \hat{\boldsymbol{H}}\left(\boldsymbol{r},\omega\right) = \hat{\boldsymbol{J}}\left(\boldsymbol{r},\omega\right) + \mathrm{j}\omega\varepsilon\hat{\boldsymbol{E}}\left(\boldsymbol{r},\omega\right)$$

$$\nabla \times \hat{\boldsymbol{E}}\left(\boldsymbol{r},\omega\right) = -\mathrm{j}\omega\mu\hat{\boldsymbol{H}}\left(\boldsymbol{r},\omega\right)$$

$$\nabla \cdot \hat{\boldsymbol{H}}\left(\boldsymbol{r},\omega\right) = 0$$

$$\nabla \cdot \hat{\boldsymbol{E}}\left(\boldsymbol{r},\omega\right) = \frac{\hat{\rho}\left(\boldsymbol{r},\omega\right)}{\varepsilon}$$

We assume linearity of material relations

Wave Equation – Frequency Domain

$$\Delta \hat{\boldsymbol{A}}\left(\boldsymbol{r},\omega\right) - \mathrm{j}\omega\mu\left(\sigma + \mathrm{j}\omega\varepsilon\right)\hat{\boldsymbol{A}}\left(\boldsymbol{r},\omega\right) = -\mu\hat{\boldsymbol{J}}_{\mathrm{source}}\left(\boldsymbol{r},\omega\right)$$

Helmholtz('s) equation

Heat Balance in Time-Harmonic Steady State

Valid for general periodic steady state

Cycle mean

$$-\int_V \left\langle \boldsymbol{E} \cdot \boldsymbol{J}_{\text{source}} \right\rangle \mathrm{d}V = \oint_S \left\langle \boldsymbol{E} \times \boldsymbol{H} \right\rangle \cdot \mathrm{d}\boldsymbol{S} + \int_V \left\langle \sigma \left| \boldsymbol{E} \right|^2 \right\rangle \mathrm{d}V$$

$$-\frac{1}{2}\int_V \mathrm{Re}\left[\hat{\boldsymbol{E}} \cdot \hat{\boldsymbol{J}}^*_{\text{source}} \right] \mathrm{d}V = \frac{1}{2}\oint_S \mathrm{Re}\left[\hat{\boldsymbol{E}} \times \hat{\boldsymbol{H}}^* \right] \cdot \mathrm{d}\boldsymbol{S} + \frac{1}{2}\int_V \sigma \left| \hat{\boldsymbol{E}} \right|^2 \mathrm{d}V$$

Valid for time-harmonic steady state

Plane Wave

Unitary vector representing the direction of propagation

Electric and magnetic fields are mutually orthogonal

Electric and magnetic fields are orthogonal to propagation direction

$$\hat{\boldsymbol{E}}\left(\boldsymbol{r},\omega\right) = \boldsymbol{E}_0\left(\omega\right)\mathrm{e}^{-\mathrm{j}k\boldsymbol{n}\cdot\boldsymbol{r}}$$

$$\hat{\boldsymbol{H}}\left(\boldsymbol{r},\omega\right) = \frac{k}{\omega\mu}\left[\boldsymbol{n}\times\boldsymbol{E}_0\left(\omega\right)\right]\mathrm{e}^{-\mathrm{j}k\boldsymbol{n}\cdot\boldsymbol{r}}$$

$$\boldsymbol{n}\cdot\boldsymbol{E}_0\left(\omega\right) = 0$$

$$\boldsymbol{n}\cdot\boldsymbol{H}_0\left(\omega\right) = 0$$

$$k^2 = -\mathrm{j}\omega\mu\left(\sigma + \mathrm{j}\omega\varepsilon\right)$$

Wave-number

The simplest wave solution of Maxwell('s) equations

Plane Wave Characteristics

General isotropic material

$$k = \sqrt{-\mathrm{j}\omega\mu\left(\sigma + \mathrm{j}\omega\varepsilon\right)}$$

$$\mathrm{Re}\left[k\right] > 0;\ \mathrm{Im}\left[k\right] < 0$$

$$\lambda = \frac{2\pi}{\mathrm{Re}\left[k\right]}$$

$$v_{\mathrm{f}} = \frac{\omega}{\mathrm{Re}\left[k\right]}$$

$$Z = \frac{\omega\mu}{k}$$

$$\delta = -\frac{1}{\mathrm{Im}\left[k\right]}$$

Vacuum

$$k = \frac{\omega}{c_0}$$

$$\mathrm{Re}\left[k\right] > 0;\ \mathrm{Im}\left[k\right] = 0$$

$$\lambda = \frac{c_0}{f}$$

$$v_{\mathrm{f}} = c_0$$

$$Z = c_0\mu_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377\ \Omega$$

$$\delta \to \infty$$

Cycle Mean Power Density of a Plane Wave

Power propagation coincides with phase propagation

$$\left\langle \boldsymbol{E}\left(\boldsymbol{r},t\right)\times\boldsymbol{H}\left(\boldsymbol{r},t\right)\right\rangle = \frac{1}{2}\frac{\mathrm{Re}\left[k\right]}{\omega\mu}\left|\boldsymbol{E}_0\left(\omega\right)\right|^2 \mathrm{e}^{2\,\mathrm{Im}\left[k\right]\boldsymbol{n}\cdot\boldsymbol{r}}\boldsymbol{n}$$

Source Free Maxwell('s) Equations in Free Space

$$\nabla \times \boldsymbol{H}(\boldsymbol{r},t) = \sigma(t) * \boldsymbol{E}(\boldsymbol{r},t) + \varepsilon(t) * \frac{\partial \boldsymbol{E}(\boldsymbol{r},t)}{\partial t}$$

$$\nabla \times \boldsymbol{E}(\boldsymbol{r},t) = -\mu(t) * \frac{\partial \boldsymbol{H}(\boldsymbol{r},t)}{\partial t}$$

$$\nabla \cdot \boldsymbol{H}(\boldsymbol{r},t) = 0$$

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r},t) = 0$$

$$\left|\boldsymbol{k}\right|^2 = k^2 = -\mathrm{j}\omega\hat{\mu}(\omega)\left(\hat{\sigma}(\omega) + \mathrm{j}\omega\hat{\varepsilon}(\omega)\right)$$

$$\hat{\boldsymbol{E}}(\boldsymbol{k},\omega) = \frac{\boldsymbol{k}}{\omega\hat{\varepsilon}(\omega) - \mathrm{j}\hat{\sigma}(\omega)} \times \hat{\boldsymbol{H}}(\boldsymbol{k},\omega)$$

$$\hat{\boldsymbol{H}}(\boldsymbol{k},\omega) = -\frac{\boldsymbol{k}}{\omega\hat{\mu}(\omega)} \times \hat{\boldsymbol{E}}(\boldsymbol{k},\omega)$$

$$\boldsymbol{k} \cdot \hat{\boldsymbol{E}}(\boldsymbol{k},\omega) = 0$$

$$\boldsymbol{k} \cdot \hat{\boldsymbol{H}}(\boldsymbol{k},\omega) = 0$$

$$\boldsymbol{F}(\boldsymbol{r},t) = \frac{1}{(2\pi)^4} \int\limits_{\boldsymbol{k},\omega} \hat{\boldsymbol{F}}(\boldsymbol{k},\omega)\, \mathrm{e}^{\mathrm{j}(\boldsymbol{k}\cdot\boldsymbol{r}+\omega t)}\, \mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega$$

Fourier's transform leads to simple algebraic equations

Spatial Wave Packet

$$\left|\boldsymbol{k}\right|^2 = k^2 = -\mathrm{j}\omega\hat{\mu}\left(\omega\right)\left(\hat{\sigma}\left(\omega\right) + \mathrm{j}\omega\hat{\varepsilon}\left(\omega\right)\right) \quad \Rightarrow \quad \omega = \omega\left(\left|\boldsymbol{k}\right|\right)$$

This can be electric or magnetic intensity

$$\boldsymbol{F}\left(\boldsymbol{r}, t\right) = \frac{1}{\left(2\pi\right)^3}\int\limits_{\boldsymbol{k}} \hat{\boldsymbol{F}}_0\left(\boldsymbol{k}\right)\mathrm{e}^{\mathrm{j}\left(\boldsymbol{k}\cdot\boldsymbol{r} + \omega\left(\left|\mathbf{k}\right|\right)t\right)}\mathrm{d}\boldsymbol{k}$$

$$\boldsymbol{k}\cdot\hat{\boldsymbol{F}}_0\left(\boldsymbol{k}\right) = 0$$

General solution to free-space Maxwell's equations

Spatial Wave Packet in Vacuum

$$\omega\left(\left|\boldsymbol{k}\right|\right) = \pm c_0 \left|\boldsymbol{k}\right|$$

$$\boldsymbol{k} \cdot \hat{\boldsymbol{F}}_0^{+}\left(\boldsymbol{k}\right) = \boldsymbol{k} \cdot \hat{\boldsymbol{F}}_0^{-}\left(\boldsymbol{k}\right) = 0$$

$$\hat{\boldsymbol{F}}_0^{-}\left(\boldsymbol{k}\right) = \left[\hat{\boldsymbol{F}}_0^{+}\left(-\boldsymbol{k}\right)\right]^*$$

$$\hat{\boldsymbol{F}}_0^{+}\left(\boldsymbol{k}\right) = \left[\hat{\boldsymbol{F}}_0^{-}\left(-\boldsymbol{k}\right)\right]^*$$

$$\boldsymbol{F}\left(\boldsymbol{r},t\right) = \frac{1}{\left(2\pi\right)^3}\int_{\boldsymbol{k}} \mathrm{e}^{\mathrm{j}\boldsymbol{k}\cdot\boldsymbol{r}}\left[\hat{\boldsymbol{F}}_0^{+}\left(\boldsymbol{k}\right)\mathrm{e}^{\mathrm{j}c_0 t\left|\boldsymbol{k}\right|} + \hat{\boldsymbol{F}}_0^{-}\left(\boldsymbol{k}\right)\mathrm{e}^{-\mathrm{j}c_0 t\left|\boldsymbol{k}\right|}\right]\mathrm{d}\boldsymbol{k}$$

$$\hat{\boldsymbol{F}}_0^{+}\left(\boldsymbol{k}\right) = \frac{1}{2}\int_{\boldsymbol{r}}\left[\boldsymbol{F}\left(\boldsymbol{r},0\right) + \frac{1}{\mathrm{j}c_0\left|\boldsymbol{k}\right|}\left.\frac{\partial \mathbf{F}\left(\boldsymbol{r},t\right)}{\partial t}\right|_{t=0}\right]\mathrm{e}^{-\mathrm{j}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{r}$$

$$\hat{\boldsymbol{F}}_0^{-}\left(\boldsymbol{k}\right) = \frac{1}{2}\int_{\boldsymbol{r}}\left[\boldsymbol{F}\left(\boldsymbol{r},0\right) - \frac{1}{\mathrm{j}c_0\left|\boldsymbol{k}\right|}\left.\frac{\partial \mathbf{F}\left(\boldsymbol{r},t\right)}{\partial t}\right|_{t=0}\right]\mathrm{e}^{-\mathrm{j}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{r}$$

The field is uniquely given by initial conditions

Spatial Wave Packet in Vacuum

$$\omega\left(\left|\boldsymbol{k}\right|\right) = \pm c_0 \left|\boldsymbol{k}\right|$$

$$\boldsymbol{E}\left(\boldsymbol{r},t\right) = \frac{1}{\left(2\pi\right)^3}\int\limits_{\mathrm{k}} \mathrm{e}^{\mathrm{j}\boldsymbol{k}\cdot\boldsymbol{r}}\left[\hat{\boldsymbol{E}}^{+}\left(\boldsymbol{k}\right)\mathrm{e}^{\mathrm{j}c_0 t\left|\boldsymbol{k}\right|} + \hat{\boldsymbol{E}}^{-}\left(\boldsymbol{k}\right)\mathrm{e}^{-\mathrm{j}c_0 t\left|\boldsymbol{k}\right|}\right]\mathrm{d}\boldsymbol{k}$$

$$\boldsymbol{H}\left(\boldsymbol{r},t\right) = -\frac{1}{\left(2\pi\right)^3}\int\limits_{\mathrm{k}} \mathrm{e}^{\mathrm{j}\boldsymbol{k}\cdot\boldsymbol{r}}\frac{\boldsymbol{k}}{Z_0\left|\boldsymbol{k}\right|}\times\left[\hat{\boldsymbol{E}}^{+}\left(\boldsymbol{k}\right)\mathrm{e}^{\mathrm{j}c_0 t\left|\boldsymbol{k}\right|} - \hat{\boldsymbol{E}}^{-}\left(\boldsymbol{k}\right)\mathrm{e}^{-\mathrm{j}c_0 t\left|\boldsymbol{k}\right|}\right]\mathrm{d}\boldsymbol{k}$$

$$\boldsymbol{k}\cdot\hat{\boldsymbol{E}}^{+}\left(\boldsymbol{k}\right) = \boldsymbol{k}\cdot\hat{\boldsymbol{E}}^{-}\left(\boldsymbol{k}\right) = 0$$

Electric and magnetic field are not independent

Vacuum Dispersion

$$\hat{\boldsymbol{E}}^{+}\left(\boldsymbol{k}\right) = \hat{\mathbf{E}}^{-}\left(\mathbf{k}\right) = \left(2\pi\right)^{2} \hat{\boldsymbol{E}}^{+}\left(k_{z}\right)\delta\left(k_{x}\right)\delta\left(k_{y}\right)$$

$$\mathbf{z}_{0} \cdot \hat{\boldsymbol{E}}^{+}\left(k_{z}\right) = \mathbf{z}_{0} \cdot \hat{\boldsymbol{E}}^{-}\left(k_{z}\right) = 0$$

1D problem

Propagation in one direction

no dispersion

$$\boldsymbol{E}\left(z,t\right) = \boldsymbol{E}^{+}\left(z + c_{0}t\right) + \boldsymbol{E}^{-}\left(z - c_{0}t\right)$$

$$\boldsymbol{H}\left(z,t\right) = -\frac{1}{Z_{0}}\mathbf{z}_{0} \times \left[\boldsymbol{E}^{+}\left(z + c_{0}t\right) - \boldsymbol{E}^{-}\left(z - c_{0}t\right)\right]$$

1D waves in vacuum propagate without dispersion

Vacuum Dispersion

In general this term does not represent translation

$$\left(\left[x,y,z\right] \pm c_0 t\right)$$

$$\boldsymbol{E}\left(\boldsymbol{r},t\right) = \frac{1}{\left(2\pi\right)^3}\int\limits_{\mathrm{k}} \mathrm{e}^{\mathrm{j}\boldsymbol{k}\cdot\boldsymbol{r}}\left[\hat{\boldsymbol{E}}^{+}\left(\boldsymbol{k}\right)\mathrm{e}^{\mathrm{j}c_0 t\left|\boldsymbol{k}\right|} + \hat{\boldsymbol{E}}^{-}\left(\boldsymbol{k}\right)\mathrm{e}^{-\mathrm{j}c_0 t\left|\boldsymbol{k}\right|}\right]\mathrm{d}\boldsymbol{k}$$

Waves propagating in all directions

2D and 3D waves in vacuum always disperse = change shape in time

Angular Spectrum Representation

$$\mathrm{Im}\left[k_z\right] < 0$$

$$\left|\boldsymbol{k}\right|^2 = k^2 = -\mathrm{j}\omega\hat{\mu}\left(\omega\right)\left(\hat{\sigma}\left(\omega\right) + \mathrm{j}\omega\hat{\varepsilon}\left(\omega\right)\right) \Rightarrow k_z = \pm\sqrt{k^2 - k_x^2 - k_y^2}$$

$$\hat{\mathbf{H}}_0\left(k_x, k_y, \omega\right) = -\frac{\mathbf{k}}{Z\left|\mathbf{k}\right|} \times \hat{\mathbf{E}}_0\left(k_x, k_y, \omega\right)$$

$$\hat{\boldsymbol{E}}_0\left(k_x, k_y, \omega\right) = \mathcal{F}_{x,y,t}\left\{\boldsymbol{E}\left(x, y, 0, t\right)\right\}$$

$$\boldsymbol{E}\left(x, y, z < 0, t\right) = \frac{1}{\left(2\pi\right)^3} \int\limits_{k_x, k_y, \omega} \mathrm{e}^{\mathrm{j}\left(k_x x + k_y y + \omega t\right)} \hat{\boldsymbol{E}}_0\left(k_x, k_y, \omega\right) \mathrm{e}^{\mathrm{j}\sqrt{k^2 - k_x^2 - k_y^2}\, z} \mathrm{d}k_x \mathrm{d}k_y \mathrm{d}\omega$$

$$\boldsymbol{E}\left(x, y, z > 0, t\right) = \frac{1}{\left(2\pi\right)^3} \int\limits_{k_x, k_y, \omega} \mathrm{e}^{\mathrm{j}\left(k_x x + k_y y + \omega t\right)} \hat{\boldsymbol{E}}_0\left(k_x, k_y, \omega\right) \mathrm{e}^{-\mathrm{j}\sqrt{k^2 - k_x^2 - k_y^2}\, z} \mathrm{d}k_x \mathrm{d}k_y \mathrm{d}\omega$$

$$\boldsymbol{k} \cdot \hat{\boldsymbol{E}}_0 = 0$$

General solution to free-space Maxwell's equations

Propagating vs Evanescent Waves

$$k_x^2 + k_y^2 < k^2$$

These waves propagate and can carry information to far distances

$$k_x^2 + k_y^2 > k^2$$

These waves exponentially decay in amplitude and cannot carry information to far distances

Field picture losses it resolution with distance from the source plane

Paraxial Waves

$$\hat{\boldsymbol{E}}_0\left(k_x,k_y,\omega\right) \Rightarrow k_x^2+k_y^2 \ll k^2 \Rightarrow \sqrt{k^2-k_x^2-k_y^2} \approx k-\frac{1}{2k}\left(k_x^2+k_y^2\right)$$

$$\boldsymbol{E}\left(x,y,z>0,t\right)=\frac{1}{\left(2\pi\right)^3}\int\limits_{k_x,k_y,\omega} \mathrm{e}^{\mathrm{j}\left(k_x x+k_y y-kz+\omega t\right)}\hat{\boldsymbol{E}}_0\left(k_x,k_y,\omega\right)\mathrm{e}^{\mathrm{j}\frac{1}{2k}\left(k_x^2+k_y^2\right)z}\,\mathrm{d}k_x\,\mathrm{d}k_y\,\mathrm{d}\omega$$

$$\boldsymbol{k}\cdot\hat{\boldsymbol{E}}_0=0$$

Propagates almost as a planewave

$$\mathbf{z}_0\cdot\hat{\boldsymbol{E}}_0\approx 0$$

Gaussian Beam

$$\hat{\boldsymbol{E}}_{0\perp}\left(k_x,k_y,\omega\right)=\mathbf{A}_{0\perp}\pi w_0^2 e^{-\frac{1}{4}w_0^2\left(k_x^2+k_y^2\right)}$$

$$\mathbf{E}_{\perp}\left(x,y,z>0,t\right)=\frac{1}{2\pi}\int_{\omega}\mathbf{A}_{0\perp}\frac{w_0}{w\left(z\right)}\mathrm{e}^{-\frac{x^2+y^2}{w^2\left(z\right)}}\mathrm{e}^{\mathrm{j}\arctan\left(\frac{z}{z_R}\right)}\mathrm{e}^{-\mathrm{j}\frac{x^2+y^2}{w^2\left(z\right)}\left[\frac{z}{z_R}\right]}\mathrm{e}^{\mathrm{j}\omega\left[t-\frac{z}{c_0}\right]}\mathrm{d}\omega$$

Gaussian profile in amplitude

Paraxial corrections

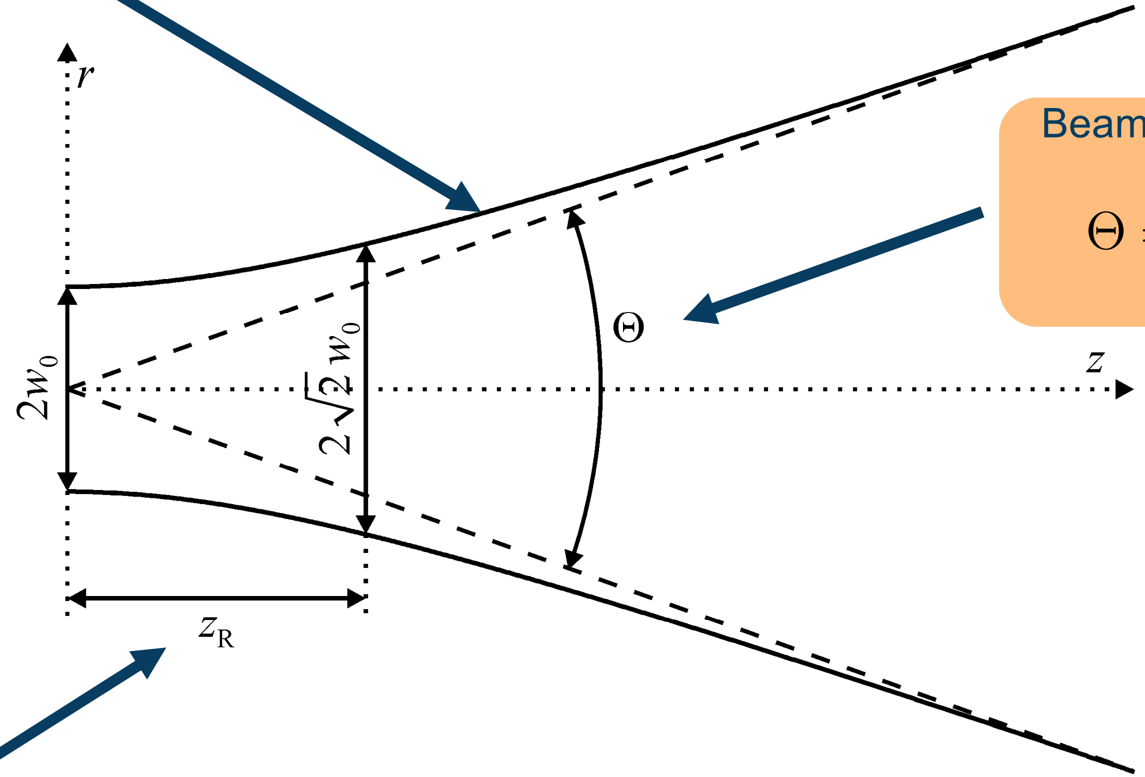
Planewave-like propagation

Approximates radiation of sources large in comparison to wavelength

Gaussian Beam

Half-width of the beam

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2}$$



Beam divergence

$$\Theta = \frac{2\lambda}{\pi w_0}$$

Rayleigh's distance

$$z_R = \frac{1}{2} k w_0^2 = \frac{\pi w_0^2}{\lambda}$$

Gaussian Beam – Time-Harmonic Case

$$\langle \boldsymbol{S} \rangle = \frac{1}{2}\mathrm{Re}\left[\hat{\boldsymbol{E}}\left(x,y,z,\omega\right)\times\hat{\boldsymbol{H}}^{*}\left(x,y,z,\omega\right)\right] = \mathbf{z}_0 S_0 \frac{w_0^2}{w^2\left(z\right)} e^{-\frac{2\rho^2}{w^2\left(z\right)}}$$

86.5 % of power flows through the beam width

Power density at origin

$$w\left(z\right) = w_0\sqrt{1+\left(\frac{z}{z_R}\right)^2}$$

Material Dispersion

Causality requirement

$$\varepsilon\left(\tau\right) = 0, \tau < 0$$

Stability requirement

$$\varepsilon\left(\tau\right) \to 0, \tau \to \infty$$

$$\boldsymbol{D}\left(\boldsymbol{r},t\right) = \int\limits_{-\infty}^{\infty} \varepsilon\left(\tau\right) \boldsymbol{E}\left(\boldsymbol{r},t-\tau\right) \mathrm{d}\tau$$

$$\boldsymbol{B}\left(\boldsymbol{r},t\right) = \int\limits_{-\infty}^{\infty} \mu\left(\tau\right) \boldsymbol{H}\left(\boldsymbol{r},t-\tau\right) \mathrm{d}\tau$$

$$\boldsymbol{J}\left(\boldsymbol{r},t\right) = \int\limits_{-\infty}^{\infty} \sigma\left(\tau\right) \boldsymbol{E}\left(\boldsymbol{r},t-\tau\right) \mathrm{d}\tau$$

$$\hat{\boldsymbol{D}}\left(\boldsymbol{r},\omega\right) = \hat{\varepsilon}\left(\omega\right) \hat{\boldsymbol{E}}\left(\boldsymbol{r},\omega\right)$$

$$\hat{\boldsymbol{B}}\left(\boldsymbol{r},\omega\right) = \hat{\mu}\left(\omega\right) \hat{\boldsymbol{H}}\left(\boldsymbol{r},\omega\right)$$

$$\hat{\boldsymbol{J}}\left(\boldsymbol{r},\omega\right) = \hat{\sigma}\left(\omega\right) \hat{\boldsymbol{E}}\left(\boldsymbol{r},\omega\right)$$

Even single planewave undergoes time dispersion when materials are present

Lorentz's Dispersion Model

Losses | Resonance | Coupling strength

$$\frac{\partial^2 \boldsymbol{P}(t)}{\partial t^2} + \Gamma \frac{\partial \boldsymbol{P}(t)}{\partial t} + \omega_0^2 \boldsymbol{P}(t) = \varepsilon_0 \omega_{\mathrm{p}}^2 \boldsymbol{E}(t)$$

$$\varepsilon(\omega) = \varepsilon_0 \left(1 + \frac{\omega_{\mathrm{p}}^2}{\omega_0^2 - \omega^2 + \mathrm{j}\omega\Gamma}\right)$$

$$\varepsilon(\omega) = \varepsilon_0 \left(1 + \sum_i \frac{\omega_{\mathrm{p},i}^2}{\omega_{0,i}^2 - \omega^2 + \mathrm{j}\omega\Gamma_i}\right)$$

Dispersion model able to describe vast amount of natural materials

Drude‘s Dispersion Model

Special case of
Lorentz‘s dispersion

$$\omega_0 = 0$$

$$\omega_\mathrm{p}^2 = \frac{\sigma_0 \Gamma}{\varepsilon_0}$$

Permittivity model

$$\varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{\omega_\mathrm{p}^2}{\omega(\omega - \mathrm{j}\Gamma)}\right)$$

$\Leftrightarrow$

Conductivity model

$$\sigma(\omega) = \frac{\sigma_0}{1 + \mathrm{j}\dfrac{\omega}{\Gamma}}$$

Collisionless plasma

$$\frac{\Gamma}{\omega} \ll 1 \Rightarrow \varepsilon(\omega) \approx \varepsilon_0 \left(1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\right)$$

Dispersion model describing neutral plasma

Appleton‘s Dispersion Model

$$\frac{\partial^2 \boldsymbol{P}(t)}{\partial t^2} + \omega_\mathrm{c} \frac{\partial \boldsymbol{P}(t)}{\partial t} \times \mathbf{z}_0 = \varepsilon_0 \omega_\mathrm{p}^2 \boldsymbol{E}(t)$$

Cyclotron frequency

Direction of magnetization

Plasma frequency

$\hat{\varepsilon} \neq \hat{\varepsilon}^\mathrm{T}$

Propagation in opposite directions is not the same

$$\hat{\varepsilon} = \varepsilon_0 \begin{bmatrix} 1 - \dfrac{\omega_\mathrm{p}^2}{\omega^2 - \omega_\mathrm{c}^2} & \dfrac{-\mathrm{j}\omega_\mathrm{c}\omega_\mathrm{p}^2}{\omega\left(\omega^2 - \omega_\mathrm{c}^2\right)} & 0 \\ \dfrac{\mathrm{j}\omega_\mathrm{c}\omega_\mathrm{p}^2}{\omega\left(\omega^2 - \omega_\mathrm{c}^2\right)} & 1 - \dfrac{\omega_\mathrm{p}^2}{\omega^2 - \omega_\mathrm{c}^2} & 0 \\ 0 & 0 & 1 - \dfrac{\omega_\mathrm{p}^2}{\omega^2} \end{bmatrix}$$

Dispersion model describing magnetized neutral plasma

Propagation in Appleton's Dispersion Model

$$\hat{\boldsymbol{E}} = \boldsymbol{E}_0 e^{jk_z z}$$

$$\boldsymbol{k} \cdot \boldsymbol{E}_0 = 0$$

Planewave propagation along magnetization

$$\frac{k_z^2}{k_0^2} = 1 - \frac{\omega_p^2}{\omega\left(\omega \pm \omega_c\right)}$$

$$\hat{E}_x = \mp j \hat{E}_y$$

Fundamental modes are circularly polarized waves

Dispersion model describing magnetized neutral plasma

Radiation

Microscopic charge velocity

Macroscopic current density

$$\frac{\partial \boldsymbol{v}(t)}{\partial t} \neq 0 \quad \Longleftrightarrow \quad \frac{\partial \boldsymbol{J}(\boldsymbol{r},t)}{\partial t} \neq 0$$

$$\Downarrow$$

$$\int_{-\infty}^{\infty} \oint_S \left(\boldsymbol{E}(\boldsymbol{r},t) \times \boldsymbol{H}(\boldsymbol{r},t)\right) \cdot \mathrm{d}\boldsymbol{S}\mathrm{d}t \neq 0$$

Any surface circumscribing the sources

Only accelerating charges can radiate

Time-Harmonic Electric Dipole

$$\hat{\boldsymbol{P}}(\boldsymbol{r},\omega) = \mathbf{z}_0 p_z(\omega)\delta(x)\delta(y)\delta(z)$$

$$\rho(\boldsymbol{r},\omega) \approx 0$$

$$\hat{\boldsymbol{A}}(\boldsymbol{r},\omega) = \mathrm{j}Z_0 k^2 \left(\boldsymbol{r}_0 \cos\theta - \boldsymbol{\theta}_0 \sin\theta\right) p_z(\omega)\frac{\mathrm{e}^{-\mathrm{j}kr}}{4\pi kr}$$

$$\hat{\boldsymbol{H}}(\boldsymbol{r},\omega) = c_0 k^3 \boldsymbol{\varphi}_0 \sin\theta\left(-1 + \frac{\mathrm{j}}{kr}\right) p_z(\omega)\frac{\mathrm{e}^{-\mathrm{j}kr}}{4\pi kr}$$

$$\hat{\boldsymbol{E}}(\boldsymbol{r},\omega) = Z_0 c_0 k^3 \left[2\boldsymbol{r}_0 \cos\theta\left(\frac{\mathrm{j}}{kr} + \frac{1}{k^2r^2}\right) + \boldsymbol{\theta}_0\left(-1 + \frac{\mathrm{j}}{kr} + \frac{1}{k^2r^2}\right)\sin\theta\right] p_z(\omega)\frac{\mathrm{e}^{-\mathrm{j}kr}}{4\pi kr}$$

Elementary source of radiation

Time-Harmonic Electric Dipole - Field Zones

Static zone

$$\hat{\boldsymbol{E}}\left(\boldsymbol{r},\omega\right) = Z_0 c_0 k^3 \left[2\boldsymbol{r}_0 \cos\theta \left(\frac{\mathrm{j}}{kr} + \frac{1}{k^2 r^2}\right) + \boldsymbol{\theta}_0 \left(-1 + \frac{\mathrm{j}}{kr} + \frac{1}{k^2 r^2}\right) \sin\theta\right] p_z\left(\omega\right) \frac{\mathrm{e}^{-\mathrm{j}kr}}{4\pi kr}$$

Induction zone

Radiation zone

Static, quasi-static and fully dynamic terms all appear in the formula

Time-Harmonic Electric Dipole - Radiation Zone

$$\hat{\boldsymbol{P}}(\boldsymbol{r},\omega) = \mathbf{z}_0 p_z(\omega)\delta(x)\delta(y)\delta(z)$$

$$\hat{\boldsymbol{E}}_\infty(\boldsymbol{r},\omega) \approx -Z_0 c_0 k^3 \boldsymbol{\theta}_0 p_z(\omega)\frac{\mathrm{e}^{-\mathrm{j}kr}}{4\pi kr}\sin\theta$$

$$\hat{\boldsymbol{H}}_\infty(\boldsymbol{r},\omega) \approx \frac{1}{Z_0}\boldsymbol{r}_0 \times \hat{\boldsymbol{E}}_\infty(\boldsymbol{r},\omega)$$

$$\langle \boldsymbol{S}_\infty \rangle = \frac{1}{2}\mathrm{Re}\left[\hat{\boldsymbol{E}}_\infty \times \hat{\boldsymbol{H}}_\infty^*\right] = \frac{1}{2Z_0}\left|\hat{\boldsymbol{E}}_\infty(\boldsymbol{r},\omega)\right|^2 \boldsymbol{r}_0$$

$$\boldsymbol{r}_0 \cdot \hat{\boldsymbol{E}}_\infty \approx 0$$

Radiated power [W]

$$P_{\mathrm{rad}} = \frac{c_0^2 Z_0 k^4}{12\pi}\left|p_z(\omega)\right|^2$$

Farfield has a planewave-like geometry

Time-Harmonic Electric Dipole – General Case

$$\hat{\boldsymbol{P}}\left(\boldsymbol{r},\omega\right)=\hat{\boldsymbol{p}}\left(\omega\right)\delta\left(\boldsymbol{r}-\boldsymbol{r}'\right)$$

$$\boldsymbol{R}=\boldsymbol{r}-\boldsymbol{r}'$$

$$\hat{\boldsymbol{H}}\left(\boldsymbol{r},\omega\right)=c_0k^3\left(\frac{\boldsymbol{R}}{R}\times\hat{\boldsymbol{p}}\right)\left(1+\frac{1}{\mathrm{j}kR}\right)\frac{\mathrm{e}^{-\mathrm{j}kR}}{4\pi kR}$$

$$\hat{\boldsymbol{E}}\left(\boldsymbol{r},\omega\right)=Z_0c_0k^3\left[-\frac{\boldsymbol{R}}{R}\times\left(\frac{\boldsymbol{R}}{R}\times\hat{\boldsymbol{p}}\right)+\left(3\frac{\boldsymbol{R}}{R}\left[\hat{\boldsymbol{p}}\cdot\frac{\boldsymbol{R}}{R}\right]-\hat{\boldsymbol{p}}\right)\left(\frac{1}{k^2R^2}+\frac{\mathrm{j}}{kR}\right)\right]\frac{\mathrm{e}^{-\mathrm{j}kR}}{4\pi kR}$$

$$R=\left|\boldsymbol{r}-\boldsymbol{r}'\right|$$

Elementary source of radiation

General Radiator

$$\hat{\boldsymbol{J}}\left(\boldsymbol{r},\omega\right),\ \boldsymbol{J}\left(\boldsymbol{r},t\right)$$

time retardation

$$\hat{\boldsymbol{A}}\left(\boldsymbol{r},\omega\right)=\frac{\mu_0}{4\pi}\int\limits_{V'}\hat{\boldsymbol{J}}\left(\boldsymbol{r}',\omega\right)\frac{\mathrm{e}^{-\mathrm{j}k_0R}}{R}\,\mathrm{d}V'$$

$$\boldsymbol{A}\left(\boldsymbol{r},t\right)=\frac{\mu_0}{4\pi}\int\limits_{V'}\frac{\boldsymbol{J}\left(\boldsymbol{r}',t-\dfrac{R}{c_0}\right)}{R}\,\mathrm{d}V'$$

$$R=\left|\boldsymbol{r}-\boldsymbol{r}'\right|$$

Superposition of dipole fields

Field in Radiation Zone – General Case FD

$$kR \gg 1 \quad \wedge \quad r \gg r'$$

$$\hat{\boldsymbol{A}}_\infty\left(\boldsymbol{r},\omega\right) \approx \frac{\mu_0}{4\pi}\frac{\mathrm{e}^{-\mathrm{j}kr}}{r}\int\limits_{V'}\hat{\boldsymbol{J}}\left(\boldsymbol{r}',\omega\right)\mathrm{e}^{\mathrm{j}k_0\mathbf{r}_0\cdot\boldsymbol{r}'}\mathrm{d}\,V'$$

$$\hat{\boldsymbol{H}}_\infty\left(\boldsymbol{r},\omega\right) \approx -\frac{\mathrm{j}\omega}{Z_0}\mathbf{r}_0 \times \hat{\boldsymbol{A}}_\infty\left(\boldsymbol{r},\omega\right)$$

$$\hat{\boldsymbol{E}}_\infty \approx -Z_0\left(\mathbf{r}_0 \times \hat{\boldsymbol{H}}_\infty\right)$$

$$\hat{\boldsymbol{E}}_\infty\left(\boldsymbol{r},\omega\right) \approx \mathrm{j}\omega\mathbf{r}_0 \times \left(\mathbf{r}_0 \times \hat{\boldsymbol{A}}_\infty\left(\boldsymbol{r},\omega\right)\right)$$

$$\left\langle \boldsymbol{S}_\infty \right\rangle = \frac{1}{2Z_0}\omega^2\left|\mathbf{r}_0 \times \hat{\boldsymbol{A}}_\infty\left(\boldsymbol{r},\omega\right)\right|^2\mathbf{r}_0$$

Farfield has a planewave-like geometry

Field in Radiation Zone – General Case TD

$$\boldsymbol{E}_\infty \approx -Z_0\left(\mathbf{r}_0 \times \boldsymbol{H}_\infty\right)$$

$$\boldsymbol{A}_\infty\left(\boldsymbol{r},t\right) \approx \frac{\mu}{4\pi r}\int\limits_{V'} \boldsymbol{J}\left(\boldsymbol{r}', t - \frac{r}{c_0} + \frac{\mathbf{r}_0 \cdot \boldsymbol{r}'}{c_0}\right)\mathrm{d}V'$$

$$\boldsymbol{H}_\infty\left(\boldsymbol{r},t\right) \approx -\frac{1}{Z_0}\mathbf{r}_0 \times \dot{\mathbf{A}}_\infty\left(\boldsymbol{r},t\right)$$

$$\boldsymbol{E}_\infty\left(\boldsymbol{r},t\right) \approx \mathbf{r}_0 \times \left(\mathbf{r}_0 \times \dot{\mathbf{A}}_\infty\left(\boldsymbol{r},t\right)\right)$$

$$\boldsymbol{S}_\infty \approx \frac{1}{Z_0}\left|\mathbf{r}_0 \times \dot{\boldsymbol{A}}_\infty\left(\boldsymbol{r},t\right)\right|^2 \mathbf{r}_0$$

Farfield has a planewave-like geometry

Radiation Zone = Rays

$$\hat{\boldsymbol{A}}_{\infty}\left(\boldsymbol{r},\omega\right) \approx \frac{\mu_0}{4\pi}\frac{\mathrm{e}^{-\mathrm{j}kr}}{r}\int_{V'}\hat{\boldsymbol{J}}\left(\boldsymbol{r}',\omega\right)\mathrm{e}^{\mathrm{j}k_0\mathbf{r}_0\cdot\boldsymbol{r}'}\,\mathrm{d}V'$$

4D Fourier(´s) transform

$$\hat{\boldsymbol{J}}\left(k_x,k_y,k_z,\omega\right) = \mathcal{F}_{x,y,z,t}\left\{\boldsymbol{J}\left(x,y,z,t\right)\right\}$$

$$\boldsymbol{A}_{\infty}\left(\boldsymbol{r},t\right) \approx \mathcal{F}_{\omega}^{-1}\left[\frac{\mu_0}{4\pi}\hat{\boldsymbol{J}}\left(-k_0\mathbf{r}_0,\omega\right)\frac{\mathrm{e}^{-\mathrm{j}k_0 r}}{r}\right]$$

Radiation diagram is formed by Fourier(`s) transform of sources

Angular Spectrum Representation (Sources)

4D Fourier(´s) transform

$$\hat{\boldsymbol{J}}\left(k_x, k_y, k_z, \omega\right) = \mathcal{F}_{x,y,z,t}\left\{\boldsymbol{J}\left(x, y, z, t\right)\right\}$$

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2}$$

$$\operatorname{Im}\left[k_z\right] < 0$$

$$\hat{\boldsymbol{G}} = \frac{\hat{\boldsymbol{J}}\left(k_x, k_y, \mp k_z, \omega\right)}{2k_z} \mathrm{e}^{\mp \mathrm{j} k_z z}$$

Valid only outside the source region

$$\boldsymbol{H}\left(x, y, \begin{matrix} z > \max\left(z'\right) \\ z < \min\left(z'\right) \end{matrix}, t\right) = \mathcal{F}^{-1}_{k_x, k_y, \omega}\left\{\left[k_x, k_y, \mp k_z\right] \times \hat{\boldsymbol{G}}\right\}$$

$$\boldsymbol{E}\left(x, y, \begin{matrix} z > \max\left(z'\right) \\ z < \min\left(z'\right) \end{matrix}, t\right) = \mathcal{F}^{-1}_{k_x, k_y, \omega}\left\{\frac{Z_0}{k}\left[k_x, k_y, \mp k_z\right] \times \left(\left[k_x, k_y, \mp k_z\right] \times \hat{\boldsymbol{G}}\right)\right\}$$

General solution to free-space Maxwell's equations

Angular Spectrum in Radiation Zone

$$\boldsymbol{F}\left(x,y,z>0,\omega\right)=\mathcal{F}^{-1}_{k_x,k_y}\left\{\hat{\boldsymbol{L}}\left(k_x,k_y\right)\mathrm{e}^{-\mathrm{j}k_z z}\right\}$$

$k_0 r \to \infty$

$$k_z=\sqrt{k_0^2-k_x^2-k_y^2}$$

$$\mathrm{Im}\left[k_z\right]<0$$

$$\boldsymbol{r}_0=\frac{\left[x,y,z\right]}{r}$$

$$\boldsymbol{F}_\infty\left(\boldsymbol{r},\omega\right)=\frac{1}{\left(2\pi\right)^2}\int\limits_{k^2>k_x^2+k_y^2}\hat{\boldsymbol{L}}\left(k_x,k_y\right)\mathrm{e}^{\mathrm{j}k_0 r\left(\frac{k_x}{k}r_{0x}+\frac{k_y}{k}r_{0y}-\frac{k_z}{k}r_{0z}\right)}\mathrm{d}k_x\mathrm{d}k_y$$

$k_0 r \to \infty$

Stationary phase method

$$\boldsymbol{F}_\infty\left(\boldsymbol{r},\omega\right)=\frac{\mathrm{j}k_0 r_{0z}}{2\pi}\hat{\boldsymbol{L}}\left(-k_0 r_{0x},-k_0 r_{0y}\right)\frac{\mathrm{e}^{-\mathrm{j}k_0 r}}{r}$$

Farfield is made of propagating planewaves

Angular Spectrum in Radiation Zone

4D Fourier(´s) transform

$$\hat{\boldsymbol{J}}\left(k_x, k_y, k_z, \omega\right) = \mathcal{F}_{x,y,z,t}\left\{\boldsymbol{J}\left(x, y, z, t\right)\right\}$$

$$\boldsymbol{H}_\infty\left(\boldsymbol{r}, t\right) \approx \mathcal{F}_\omega^{-1}\left\{-\frac{\mathrm{j}k_0}{4\pi}\boldsymbol{r}_0 \times \hat{\boldsymbol{J}}\left(-k_0\boldsymbol{r}_0, \omega\right)\frac{\mathrm{e}^{-\mathrm{j}k_0 r}}{r}\right\}$$

$$\boldsymbol{E}_\infty\left(\boldsymbol{r}, t\right) \approx \mathcal{F}_\omega^{-1}\left\{\frac{\mathrm{j}k_0 Z_0}{4\pi}\boldsymbol{r}_0 \times \left[\boldsymbol{r}_0 \times \hat{\boldsymbol{J}}\left(-k_0\boldsymbol{r}_0, \omega\right)\right]\frac{\mathrm{e}^{-\mathrm{j}k_0 r}}{r}\right\}$$

$$\boldsymbol{r}_0 = \frac{\left[x, y, z\right]}{r}$$

Farfield is made of propagating planewaves

Planar Material Boundary

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2}$$

Incident wave

Reflected (1 → 1) / Transmitted (2 → 1) wave

$$\boldsymbol{H}\left(z<0\right) = \mathcal{F}^{-1}_{k_x,k_y}\left\{\boldsymbol{H}_1^+\left(k_x,k_y,\omega\right)\mathrm{e}^{-\mathrm{j}k_{z1}z} + \boldsymbol{H}_1^-\left(k_x,k_y,\omega\right)\mathrm{e}^{\mathrm{j}k_{z1}z}\right\}$$

$$\boldsymbol{E}\left(z<0\right) = \mathcal{F}^{-1}_{k_x,k_y}\left\{Z_1\frac{\left[k_x,k_y,-k_{z1}\right]\times\boldsymbol{H}_1^+\left(k_x,k_y,\omega\right)}{k_1}\mathrm{e}^{-\mathrm{j}k_{z1}z} + Z_1\frac{\left[k_x,k_y,k_{z1}\right]\times\boldsymbol{H}_1^-\left(k_x,k_y,\omega\right)}{k_1}\mathrm{e}^{\mathrm{j}k_{z1}z}\right\}$$

$$\boldsymbol{H}\left(z>0\right) = \mathcal{F}^{-1}_{k_x,k_y}\left\{\boldsymbol{H}_2^+\left(k_x,k_y,\omega\right)\mathrm{e}^{-\mathrm{j}k_{z2}z} + \boldsymbol{H}_2^-\left(k_x,k_y,\omega\right)\mathrm{e}^{\mathrm{j}k_{z2}z}\right\}$$

$$\boldsymbol{E}\left(z>0\right) = \mathcal{F}^{-1}_{k_x,k_y}\left\{Z_2\frac{\left[k_x,k_y,-k_{z2}\right]\times\boldsymbol{H}_2^+\left(k_x,k_y,\omega\right)}{k_2}\mathrm{e}^{-\mathrm{j}k_{z2}z} + Z_2\frac{\left[k_x,k_y,k_{z2}\right]\times\boldsymbol{H}_2^-\left(k_x,k_y,\omega\right)}{k_2}\mathrm{e}^{\mathrm{j}k_{z2}z}\right\}$$

Boundary is at $z = 0$

Reflected (2 → 2) / Transmitted (1 → 2) wave

Incident wave

Field is composed of incident, reflected and transmitted waves

Planar Material Boundary – Boundary Conditions

$$\left[k_x, k_y, \mp k_{z1}\right] \cdot \mathbf{H}_1^{\pm} = 0 \qquad \left[k_x, k_y, \mp k_{z2}\right] \cdot \mathbf{H}_2^{\pm} = 0$$

$$\mathbf{z}_0 \times \boldsymbol{H}_1^{+} + \mathbf{z}_0 \times \boldsymbol{H}_1^{-} = \mathbf{z}_0 \times \boldsymbol{H}_2^{+} + \mathbf{z}_0 \times \boldsymbol{H}_2^{-}$$

$$Z_1 \frac{\mathbf{z}_0 \times \left(\left[k_x, k_y, -k_{z1}\right] \times \boldsymbol{H}_1^{+}\right)}{k_1} + Z_1 \frac{\mathbf{z}_0 \times \left(\left[k_x, k_y, k_{z1}\right] \times \boldsymbol{H}_1^{-}\right)}{k_1} =$$

$$Z_2 \frac{\mathbf{z}_0 \times \left(\left[k_x, k_y, -k_{z2}\right] \times \boldsymbol{H}_2^{+}\right)}{k_2} + Z_2 \frac{\mathbf{z}_0 \times \left(\left[k_x, k_y, k_{z2}\right] \times \boldsymbol{H}_2^{-}\right)}{k_2}$$

k_x, k_y are equal on both sides

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2}$$

$$\mathrm{Im}\left[k_z\right] < 0$$

Relations valid for both propagative and evanescent waves

Perpendicular Incidence – Matrix Form

$k_x = k_y = 0$

$$[S] = \frac{1}{T_{22}} \begin{bmatrix} -T_{21} & 1 \\ \det T & T_{12} \end{bmatrix}$$

$$[T] = \frac{1}{S_{12}} \begin{bmatrix} -\det S & S_{22} \\ -S_{11} & 1 \end{bmatrix}$$

Transmission matrix (multilayer cascade)

$$\begin{bmatrix} E_2^+ \\ E_2^- \end{bmatrix} = [T] \begin{bmatrix} E_1^+ \\ E_1^- \end{bmatrix}$$

$$\begin{bmatrix} E_1^- \\ E_2^+ \end{bmatrix} = [S] \begin{bmatrix} E_1^+ \\ E_2^- \end{bmatrix}$$

Scattering matrix (experiments)

$$[T] = \frac{1}{2Z_1} \begin{bmatrix} Z_1 + Z_2 & Z_1 - Z_2 \\ Z_1 - Z_2 & Z_1 + Z_2 \end{bmatrix}$$

$$[S] = \frac{1}{Z_2 + Z_1} \begin{bmatrix} Z_2 - Z_1 & 2Z_1 \\ 2Z_2 & Z_1 - Z_2 \end{bmatrix}$$

Matrices with the use across the electrical engineering

Perpendicular Incidence – Interesting Cases

Wavelength inside the slab

Transparent dielectric layer

$$d = \frac{\lambda_2}{2} n \quad \Rightarrow \quad T = \begin{bmatrix} (-1)^n & 0 \\ 0 & (-1)^n \end{bmatrix}, \; S = \begin{bmatrix} 0 & (-1)^n \\ (-1)^n & 0 \end{bmatrix}$$

Bragg's mirror
(dielectric mirror)

$$k_0 \left(n_1 d_1 + n_2 d_2 \right) = N\pi$$

Alternating dielectric layers

Technically important special cases

Oblique Incidence – TM / TE Case

$$k_x = k_x$$
$$k_y = 0$$

Snell‘s law of refraction

$$k_1 \sin \alpha_{\mathrm{inc}} = k_2 \sin \alpha_{\mathrm{trans}}$$

$$k_1 \sin \alpha_{\mathrm{inc}} = k_1 \sin \alpha_{\mathrm{refl}}$$

The angle of incidence equals to the angle of reflection

Snell‘s law is a property of k-vectors

Oblique Incidence – TM Case

$$k_y = 0$$
$$H_x = 0$$
$$H_z = 0$$

$$R_{1\to 1}^{\mathrm{TM}} = \frac{E_{1x}^{-}}{E_{1x}^{+}} = \frac{\dfrac{k_{z2}}{k_2} Z_2 - \dfrac{k_{z1}}{k_1} Z_1}{\dfrac{k_{z2}}{k_2} Z_2 + \dfrac{k_{z1}}{k_1} Z_1}$$

$$T_{1\to 2}^{\mathrm{TM}} = \frac{E_{2x}^{+}}{E_{1x}^{+}} = \frac{2 Z_2 \dfrac{k_{z1}}{k_1}}{\dfrac{k_{z2}}{k_2} Z_2 + \dfrac{k_{z1}}{k_1} Z_1}$$

Generalization of reflection and transmission to oblique incidence

Oblique Incidence – TM Case

$$k_y = 0$$
$$H_x = 0$$
$$H_z = 0$$

Brewster's angle

$$R_{1\to 1}^{\mathrm{TM}} = 0 \quad \Rightarrow \quad \frac{k_x}{k_1} = \sqrt{\frac{\varepsilon_2\left(\mu_2\varepsilon_1 - \mu_1\varepsilon_2\right)}{\mu_1\left(\varepsilon_1^2 - \varepsilon_2^2\right)}}$$

Vanishing reflection on a boundary

Simplification for pure dielectrics

$$\frac{k_x}{k_1} = \left(1 + \frac{\varepsilon_1}{\varepsilon_2}\right)^{-\frac{1}{2}}$$

Can be used for polarizing unpolarizaed light beams

Oblique Incidence – TE Case

$$k_y = 0$$
$$E_x = 0$$
$$E_z = 0$$

$$R_{1\to 1}^{\mathrm{TE}} = \frac{E_{1y}^{-}}{E_{1y}^{+}} = \frac{\dfrac{k_{z1}}{k_1} Z_2 - \dfrac{k_{z2}}{k_2} Z_1}{\dfrac{k_{z1}}{k_1} Z_2 + \dfrac{k_{z2}}{k_2} Z_1}$$

$$T_{1\to 2}^{\mathrm{TE}} = \frac{E_{2y}^{+}}{E_{1y}^{+}} = \frac{2 Z_2 \dfrac{k_{z1}}{k_1}}{\dfrac{k_{z1}}{k_1} Z_2 + \dfrac{k_{z2}}{k_2} Z_1}$$

Generalization of reflection and transmission to oblique incidence

Oblique Incidence – TE Case

$$k_y = 0$$
$$E_x = 0$$
$$E_z = 0$$

Brewster's angle

$$R_{1\to 1}^{\mathrm{TE}} = 0 \quad \Rightarrow \quad \frac{k_x}{k_1} = \sqrt{\frac{\mu_2\left(\varepsilon_2\mu_1 - \varepsilon_1\mu_2\right)}{\varepsilon_1\left(\mu_1^2 - \mu_2^2\right)}}$$

Vanishing reflection on a boundary

Simplification for pure magnetics

$$\frac{k_x}{k_1} = \left(1 + \frac{\mu_1}{\mu_2}\right)^{-\frac{1}{2}}$$

Unrealistic scenario for natural materials

Oblique Incidence – Total Reflection

$$\frac{k_x}{k_1} > \frac{k_2}{k_1} = \frac{n_2}{n_1} < 1$$

$$\left|R_{1\to 1}^{\mathrm{TM}}\right| = \left|R_{1\to 1}^{\mathrm{TE}}\right| = 1$$

Valid for both, the TM and the TE case

Guided TEM Wave

Wave propagation identical to a planewave

$$k^2 = -\mathrm{j}\omega\mu\left(\sigma + \mathrm{j}\omega\varepsilon\right)$$

Geometry of a planewave

$$\hat{\boldsymbol{E}}\left(\boldsymbol{r},\omega\right) = \boldsymbol{E}_{\perp}\left(x,y,\omega\right)\mathrm{e}^{-\mathrm{j}kz}$$

$$\hat{\boldsymbol{H}}\left(\boldsymbol{r},\omega\right) = \boldsymbol{H}_{\perp}\left(x,y,\omega\right)\mathrm{e}^{-\mathrm{j}kz}$$

$$\hat{\boldsymbol{H}} = \frac{k}{\omega\mu}\left(\boldsymbol{z}_0 \times \hat{\boldsymbol{E}}\right)$$

$$\Delta_{\perp}\boldsymbol{E}_{\perp} = 0$$
$$\Delta_{\perp}\boldsymbol{H}_{\perp} = 0$$

$$\boldsymbol{n} \times \boldsymbol{E}_{\perp} = 0$$

Boundary condition on the conductor

Generalization of a planewave

Circuit Parameters of the TEM Wave

$$\hat{U}(z,\omega) = \hat{U}_0(\omega)\mathrm{e}^{-\mathrm{j}kz}$$

$$\hat{I}(z,\omega) = \hat{I}_0(\omega)\mathrm{e}^{-\mathrm{j}kz}$$

Enclosing conductor

$$\hat{I}_0(\omega) = \oint_l \boldsymbol{H}_\perp \cdot \mathrm{d}\boldsymbol{l} = \frac{k}{\omega\mu} \cdot \frac{Q_{\mathrm{pul}}}{\varepsilon}$$

$$\hat{U}_0(\omega) = -\int_A^B \boldsymbol{E}_\perp \cdot \mathrm{d}\boldsymbol{l} = \frac{\omega\mu}{k} \cdot \frac{\Phi_{\mathrm{pul}}}{\mu}$$

Between conductors

$$Z_{\mathrm{TRL}} = \frac{\hat{U}_0(\omega)}{\hat{I}_0(\omega)} = \frac{\omega\mu}{k} \cdot \frac{\varepsilon}{C_{\mathrm{pul}}} = \frac{\omega\mu}{k} \cdot \frac{L_{\mathrm{pul}}}{\mu} = \sqrt{\frac{L_{\mathrm{pul}}}{C_{\mathrm{pul}}}}$$

$$v_{\mathrm{phase}} = \frac{1}{\sqrt{C_{\mathrm{pul}} L_{\mathrm{pul}}}} = \frac{1}{\sqrt{\varepsilon\mu}}$$

Per unit length

Velocity of phase propagation

The Telegraph Equations

$$\frac{\partial U(z,t)}{\partial z} = -L_{\mathrm{pul}} \frac{\partial I(z,t)}{\partial t}$$

$$\frac{\partial I(z,t)}{\partial z} = -C_{\mathrm{pul}} \frac{\partial U(z,t)}{\partial t}$$

Circuit analog of Maxwell's equations

Guided TE and TM Waves

Wave propagation differs from a planewave

$$k_z^2 = k^2 - k_\perp^2$$

$$\hat{\boldsymbol{E}}(\boldsymbol{r},\omega) = \left[\boldsymbol{E}_\perp(\boldsymbol{r}_\perp,\omega) + \boldsymbol{z}_0 E_z(\boldsymbol{r}_\perp,\omega)\right]\mathrm{e}^{-\mathrm{j}k_z z}$$

$$\hat{\boldsymbol{H}}(\boldsymbol{r},\omega) = \left[\boldsymbol{H}_\perp(\boldsymbol{r}_\perp,\omega) + \boldsymbol{z}_0 H_z(\boldsymbol{r}_\perp,\omega)\right]\mathrm{e}^{-\mathrm{j}k_z z}$$

$$\boldsymbol{E}_\perp = -\frac{1}{k_\perp^2}\left(\mathrm{j}k_z \nabla_\perp E_z - \mathrm{j}\omega\mu \boldsymbol{z}_0 \times \nabla_\perp H_z\right)$$

$$\boldsymbol{H}_\perp = -\frac{1}{k_\perp^2}\left(\mathrm{j}k_z \nabla_\perp H_z + \left(\sigma + \mathrm{j}\omega\varepsilon\right)\boldsymbol{z}_0 \times \nabla_\perp E_z\right)$$

$$\Delta_\perp E_z + k_\perp^2 E_z = 0$$

$$\Delta_\perp H_z + k_\perp^2 H_z = 0$$

TEM mode must be completed with TE and TM modes to form a complete set

PEC Waveguides – pure TE, TM modes

Impedances differ from those of a planewave

Boundary condition on the conductor

$$\boldsymbol{n} \times \hat{\boldsymbol{E}} = 0$$

$$Z^{\mathrm{TE}\,/\,\mathrm{TM}} \mathbf{H}_{\perp} = \mathbf{z}_0 \times \mathbf{E}_{\perp}$$

$$Z^{\mathrm{TE}} = \frac{\omega\mu}{k_z}, \; Z^{\mathrm{TM}} = \frac{\mathrm{j}k_z}{\sigma + \mathrm{j}\omega\varepsilon}$$

- Wavenumbers $k_{\perp} > 0$ form a discrete set
- Modes are orthogonal in waveguide cross-section
- Modes form a complete set in waveguide cross-section

TEM mode must be completed with TE and TM modes to form a complete set

PEC Waveguides – modal orthogonality

$$\int_S \boldsymbol{E}_{\perp\alpha} \cdot \boldsymbol{E}^*_{\perp\beta} \mathrm{d}S = C\delta_{\alpha\beta}$$

cross-section of the waveguide

$$\int_S \left(\boldsymbol{E}_{\perp\alpha} \times \boldsymbol{H}^*_{\perp\beta}\right) \cdot \mathbf{z}_0 \mathrm{d}S = \frac{1}{Z^*_\beta} \int_S \boldsymbol{E}_{\perp\alpha} \cdot \boldsymbol{E}^*_{\perp\beta} \mathrm{d}S$$

$$\int_S \boldsymbol{H}_{\perp\alpha} \cdot \boldsymbol{H}^*_{\perp\beta} \mathrm{d}S = \frac{1}{Z_\alpha Z^*_\beta} \int_S \boldsymbol{E}_{\perp\alpha} \cdot \boldsymbol{E}^*_{\perp\beta} \mathrm{d}S$$

Waveguide modes form an orthogonal set

PEC Waveguides – modal decomposition

positive direction

$$\hat{\boldsymbol{E}}^{+}(\boldsymbol{r},\omega)=\sum_{\alpha}C_{\alpha}^{+}\left[\boldsymbol{E}_{\perp\alpha}(\boldsymbol{r}_{\perp},\omega)+\mathbf{z}_0E_{z\alpha}(\boldsymbol{r}_{\perp},\omega)\right]\mathrm{e}^{-\mathrm{j}k_{z\alpha}z}$$

$$\hat{\boldsymbol{H}}^{+}(\boldsymbol{r},\omega)=\sum_{\alpha}C_{\alpha}^{+}\left[\boldsymbol{H}_{\perp\alpha}(\boldsymbol{r}_{\perp},\omega)+\mathbf{z}_0H_{z\alpha}(\boldsymbol{r}_{\perp},\omega)\right]\mathrm{e}^{-\mathrm{j}k_{z\alpha}z}$$

negative direction

$$\hat{\boldsymbol{E}}^{-}(\boldsymbol{r},\omega)=\sum_{\alpha}C_{\alpha}^{-}\left[\boldsymbol{E}_{\perp\alpha}(\boldsymbol{r}_{\perp},\omega)-\mathbf{z}_0E_{z\alpha}(\boldsymbol{r}_{\perp},\omega)\right]\mathrm{e}^{\mathrm{j}k_{z\alpha}z}$$

$$\hat{\boldsymbol{H}}^{-}(\boldsymbol{r},\omega)=\sum_{\alpha}C_{\alpha}^{-}\left[-\boldsymbol{H}_{\perp\alpha}(\boldsymbol{r}_{\perp},\omega)+\mathbf{z}_0H_{z\alpha}(\boldsymbol{r}_{\perp},\omega)\right]\mathrm{e}^{\mathrm{j}k_{z\alpha}z}$$

Any field within a waveguide can be composed of its modes

PEC Waveguides – Field Sources

Known field at arbitrary cross-section of the waveguide

$$C_\beta^{\pm} = \frac{1}{2}\frac{\int\limits_S \left[\boldsymbol{E}_{\perp\beta}^{*}\left(\boldsymbol{r}_{\perp},\omega\right)\cdot\hat{\boldsymbol{E}}\left(\boldsymbol{r},\omega\right) \pm \left|Z_\beta\right|^2 \boldsymbol{H}_{\perp\beta}^{*}\left(\boldsymbol{r}_{\perp},\omega\right)\cdot\hat{\boldsymbol{H}}\left(\boldsymbol{r},\omega\right)\right]\mathrm{d}S}{\int\limits_S \boldsymbol{E}_{\perp\beta}^{*}\left(\boldsymbol{r}_{\perp},\omega\right)\cdot\boldsymbol{E}_{\perp\beta}\left(\boldsymbol{r}_{\perp},\omega\right)\mathrm{d}S}\mathrm{e}^{\pm\mathrm{j}k_{z\beta}z}$$

transversal field of the waveguide mode

Tangential fields within the cross-section fully define the field everywhere

PEC Waveguides – Field Sources

Full field of the waveguide mode

Source current density existing within the waveguide

Valid to the right (+) or to the left (-) of the source region

$$C_\beta^{\pm} = -\frac{Z_\beta}{2}\frac{\int\limits_V \hat{\boldsymbol{E}}_\beta^{\mp}\left(\boldsymbol{r},\omega\right)\cdot\hat{\boldsymbol{J}}_{\mathrm{s}}\left(\boldsymbol{r},\omega\right)\mathrm{d}V}{\int\limits_S \boldsymbol{E}_{\perp\beta}\left(\boldsymbol{r}_\perp,\omega\right)\cdot\boldsymbol{E}_{\perp\beta}\left(\boldsymbol{r}_\perp,\omega\right)\mathrm{d}S}$$

transversal field of the waveguide mode

This is how waveguide modes are excited

Dielectric Waveguides – mixed TE + TM modes

Boundary condition
on dielectric interface

$$\boldsymbol{n} \times \left[\hat{\boldsymbol{E}}_1 - \hat{\boldsymbol{E}}_2 \right] = 0$$

$$\boldsymbol{n} \times \left[\hat{\boldsymbol{H}}_1 - \hat{\boldsymbol{H}}_2 \right] = 0$$

- Finite number of guided modes
- Continuum of radiating modes
- Only combination of guided and radiating modes forms a complete set in the waveguide cross-section

General field is not guided by a dielectric waveguide

Cavity Resonators

Master equation

$$\Delta \hat{\boldsymbol{E}}_n + \frac{\omega_n^2}{c_0^2} \varepsilon_{\mathrm{r}} \mu_{\mathrm{r}} \hat{\boldsymbol{E}}_n = 0$$

Nontrivial solution only exists in a lossless cavity

Boundary condition on the conductor

$$\boldsymbol{n} \times \hat{\boldsymbol{E}}_n = 0$$

Notice that field value is defined on a closed surface

- Eigenfrequencies $\omega_n > 0$ form a discrete set
- Modes are orthogonal in the volume of the cavity
- Modes form a complete set in the volume of the cavity

Field in a lossless cavity forms an exception to the uniqueness theorem

Closed Waveguide as a Cavity Resonator

Dispersion relation

$$k_{z\alpha} = \frac{p\pi}{L}$$

$$\frac{\omega_\alpha^2}{c_0^2}\varepsilon_{\mathrm{r}}\mu_{\mathrm{r}} = k_{z\alpha}^2 + k_\perp^2$$

$$\hat{\boldsymbol{E}}_\alpha\left(\boldsymbol{r},\omega\right) = \boldsymbol{E}_{\alpha\perp}\left(\boldsymbol{r}_\perp,\omega\right)\sin\left(k_{z\alpha}z\right) + \mathrm{j}\boldsymbol{z}_0 E_{z\alpha}\left(\boldsymbol{r}_\perp,\omega\right)\cos\left(k_{z\alpha}z\right)$$

$$\hat{\boldsymbol{H}}_\alpha\left(\boldsymbol{r},\omega\right) = \mathrm{j}\boldsymbol{H}_{\perp\alpha}\left(\boldsymbol{r}_\perp,\omega\right)\cos\left(k_{z\alpha}z\right) + \boldsymbol{z}_0 H_{z\alpha}\left(\boldsymbol{r}_\perp,\omega\right)\sin\left(k_{z\alpha}z\right)$$

Electric and magnetic fields
are 90° out of phase

Waveguide modes already solve the wave equation

Excitation of a PEC Cavity

Vector potential is expanded into cavity modes

$$\boldsymbol{A}(\boldsymbol{r},\omega) = \sum_{\alpha} C_{\alpha}(\omega)\boldsymbol{A}_{\alpha}(\boldsymbol{r})$$

Modes of lossless cavity

$$\Delta \boldsymbol{A}_{\alpha}(\boldsymbol{r}) + \lambda_{\alpha}^{2}\boldsymbol{A}_{\alpha}(\boldsymbol{r}) = 0$$

$$\boldsymbol{n}(\boldsymbol{r}) \times \boldsymbol{E}_{\alpha}(\boldsymbol{r}) = 0$$

$$C_{\alpha}(\omega) = -\frac{\mu(\omega)}{k^{2}(\omega) - \lambda_{\alpha}^{2}} \cdot \frac{\int_{V} \boldsymbol{A}_{\alpha}^{*}(\boldsymbol{r}) \cdot \boldsymbol{J}(\boldsymbol{r},\omega)\,\mathrm{d}V}{\int_{V} \boldsymbol{A}_{\alpha}^{*}(\boldsymbol{r}) \cdot \boldsymbol{A}_{\alpha}(\boldsymbol{r})\,\mathrm{d}V}$$

Modes of a lossless cavity are used as a basis

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