3D Computer Vision

Radim Šára   Martin Matoušek

Center for Machine Perception
Department of Cybernetics
Faculty of Electrical Engineering
Czech Technical University in Prague

https://cw.fel.cvut.cz/wiki/courses/tdv/start

http://cmp.felk.cvut.cz
mailto:sara@cmp.felk.cvut.cz
phone ext. 7203

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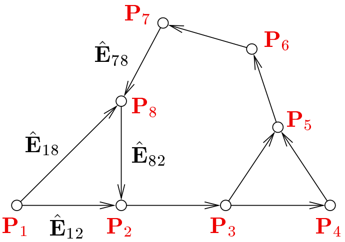
Open Informatics Master's Course

►Reconstructing Camera System from Pairs (Correspondence-Free)

Problem: Given a set of p decomposed pairwise essential matrices $\hat{\mathbf{E}}_{ij} = [\hat{\mathbf{t}}_{ij}]_\times \hat{\mathbf{R}}_{ij}$ and calibration matrices $\mathbf{K}_i$ reconstruct the camera system $\mathbf{P}_i$, $i = 1, \ldots, k$

→81 and →150 on representing $\mathbf{E}$



We construct calibrated camera pairs $\hat{\mathbf{P}}_{ij} \in \mathbb{R}^{6,4}$ see (17)

$$\hat{\mathbf{P}}_{ij} = \begin{bmatrix} \mathbf{K}_i^{-1}\hat{\mathbf{P}}_i \\ \mathbf{K}_j^{-1}\hat{\mathbf{P}}_j \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \hat{\mathbf{R}}_{ij} & \hat{\mathbf{t}}_{ij} \end{bmatrix} \in \mathbb{R}^{6,4}$$

- singletons i, j correspond to graph nodes — k nodes
- pairs ij correspond to graph edges — p edges

$\hat{\mathbf{P}}_{ij}$ are in different coordinate systems but these are related by similarities $\hat{\mathbf{P}}_{ij}\mathbf{H}_{ij} = \mathbf{P}_{ij}$

$$\underbrace{\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \hat{\mathbf{R}}_{ij} & \hat{\mathbf{t}}_{ij} \end{bmatrix}}_{\mathbb{R}^{6,4}} \underbrace{\begin{bmatrix} \mathbf{R}_{ij} & \mathbf{t}_{ij} \\ \mathbf{0}^\top & s_{ij} \end{bmatrix}}_{\mathbf{H}_{ij} \in \mathbb{R}^{4,4}} \overset{!}{=} \underbrace{\begin{bmatrix} \mathbf{R}_i & \mathbf{t}_i \\ \mathbf{R}_j & \mathbf{t}_j \end{bmatrix}}_{\mathbb{R}^{6,4}} \tag{28}$$

- (28) is a linear system of $24p$ eqs. in $7p + 6k$ unknowns — $7p \sim (\mathbf{t}_{ij}, \mathbf{R}_{ij}, s_{ij})$, $6k \sim (\mathbf{R}_i, \mathbf{t}_i)$
- each $\hat{\mathbf{P}}_i = (\mathbf{R}_i,\ \mathbf{t}_i)$ appears on the RHS as many times as is the degree of node $\mathbf{P}_i$ eg. P_5 3×

►cont'd

Eq. (28) implies $\begin{bmatrix} \mathbf{R}_{ij} \\ \hat{\mathbf{R}}_{ij}\mathbf{R}_{ij} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_i \\ \mathbf{R}_j \end{bmatrix}$ and $\begin{bmatrix} \mathbf{t}_{ij} \\ \hat{\mathbf{R}}_{ij}\mathbf{t}_{ij} + s_{ij}\hat{\mathbf{t}}_{ij} \end{bmatrix} = \begin{bmatrix} \mathbf{t}_i \\ \mathbf{t}_j \end{bmatrix}$

- $\mathbf{R}_{ij}$ and $\mathbf{t}_{ij}$ can be eliminated:

$$\hat{\mathbf{R}}_{ij}\mathbf{R}_i = \mathbf{R}_j, \qquad \hat{\mathbf{R}}_{ij}\mathbf{t}_i + s_{ij}\hat{\mathbf{t}}_{ij} = \mathbf{t}_j, \qquad s_{ij} > 0 \tag{29}$$

- note transformations that do not change these equations — assuming no error in $\hat{\mathbf{R}}_{ij}$

  1. $\mathbf{R}_i \mapsto \mathbf{R}_i\mathbf{R}$, 2. $\mathbf{t}_i \mapsto \sigma\,\mathbf{t}_i$ and $s_{ij} \mapsto \sigma s_{ij}$, 3. $\mathbf{t}_i \mapsto \mathbf{t}_i + \mathbf{R}_i\mathbf{t}$

- the global frame is fixed, e.g. by selecting

$$\mathbf{R}_1 = \mathbf{I}, \qquad \sum_{i=1}^{k} \mathbf{t}_i = \mathbf{0}, \qquad \frac{1}{p}\sum_{i,j} s_{ij} = 1 \tag{30}$$

- rotation equations are decoupled from translation equations
- in principle, s_{ij} could correct the sign of $\hat{\mathbf{t}}_{ij}$ from essential matrix decomposition →81
  but $\mathbf{R}_i$ cannot correct the α sign in $\hat{\mathbf{R}}_{ij}$

  ⇒ therefore make sure all points are in front of cameras and constrain $s_{ij} > 0$; →83

+ pairwise correspondences are sufficient
− suitable for well-distributed cameras only (dome-like configurations)

otherwise intractable or numerically unstable

Finding The Rotation Component in Eq. (29)

1. **Poor Man's Algorithm:**
   a) create a Minimum Spanning Tree of $\mathcal{G}$ from →133
   b) propagate rotations from $\mathbf{R}_1 = \mathbf{I}$ via $\hat{\mathbf{R}}_{ij}\mathbf{R}_i = \mathbf{R}_j$ from (29)

2. **Rich Man's Algorithm:**
   Consider $\hat{\mathbf{R}}_{ij}\mathbf{R}_i = \mathbf{R}_j$, $(i,j) \in E(\mathcal{G})$, where $\mathbf{R}$ are a 3×3 rotation matrices
   Errors per columns $c = 1,2,3$ of $\mathbf{R}_j$:

$$\mathbf{e}_{ij}^c = \hat{\mathbf{R}}_{ij}\mathbf{r}_i^c - \mathbf{r}_j^c, \qquad \text{for all } i,\ j$$

   Solve

$$\arg\min \sum_{(i,j)\in E(\mathcal{G})} \sum_{c=1}^{3} (\mathbf{e}_{ij}^c)^\top \mathbf{e}_{ij}^c \quad \text{s.t.} \quad (\mathbf{r}_i^k)^\top(\mathbf{r}_j^l) = \begin{cases} 1 & i = j \wedge k = l \\ 0 & i \neq j \wedge k = l \\ 0 & i = j \wedge k \neq l \end{cases}$$

   this is a quadratic programming problem

3. **SVD-Lover's Algorithm:**
   Ignore the constraints and project the solution onto rotation matrices see next

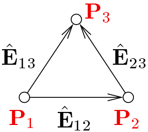
SVD Algorithm (cont'd)

Per columns $c = 1, 2, 3$ of $\mathbf{R}_j$:

$$\hat{\mathbf{R}}_{ij}\mathbf{r}_i^c - \mathbf{r}_j^c = \mathbf{0}, \qquad \text{for all } i,\ j \tag{31}$$

- fix c and denote $\mathbf{r}^c = \left[\mathbf{r}_1^c, \mathbf{r}_2^c, \ldots, \mathbf{r}_k^c\right]^\top$ c-th columns of all rotation matrices stacked; $\mathbf{r}^c \in \mathbb{R}^{3k}$
- then (31) becomes $\mathbf{D}\,\mathbf{r}^c = \mathbf{0}$ $\mathbf{D} \in \mathbb{R}^{3p,3k}$
- $3p$ equations for $3k$ unknowns $\rightarrow p \geq k$ in a 1-connected graph we have to fix $\mathbf{r}_1^c = [1, 0, 0]$

Ex: $(k = p = 3)$



$\rightarrow$

$$\begin{aligned} \hat{\mathbf{R}}_{12}\mathbf{r}_1^c - \mathbf{r}_2^c &= \mathbf{0} \\ \hat{\mathbf{R}}_{23}\mathbf{r}_2^c - \mathbf{r}_3^c &= \mathbf{0} \\ \hat{\mathbf{R}}_{13}\mathbf{r}_1^c - \mathbf{r}_3^c &= \mathbf{0} \end{aligned} \quad \rightarrow \quad \mathbf{D}\,\mathbf{r}^c = \begin{bmatrix} \hat{\mathbf{R}}_{12} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{R}}_{23} & -\mathbf{I} \\ \hat{\mathbf{R}}_{13} & \mathbf{0} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{r}_1^c \\ \mathbf{r}_2^c \\ \mathbf{r}_3^c \end{bmatrix} = \mathbf{0}$$

- must hold for any c

Idea: [Martinec & Pajdla CVPR 2007]

1. find the space of all $\mathbf{r}^c \in \mathbb{R}^{3k}$ that solve (31) $\mathbf{D}$ is sparse, use `[V,E] = eigs(D'*D,3,0);` (Matlab)
2. choose 3 unit orthogonal vectors in this space 3 smallest eigenvectors
3. find closest rotation matrices per cam. using SVD because $\|\mathbf{r}^c\| = 1$ is necessary but insufficient $\mathbf{R}_i^* = \mathbf{U}\mathbf{V}^\top$, where $\mathbf{R}_i = \mathbf{U}\mathbf{D}\mathbf{V}^\top$

- global world rotation is arbitrary

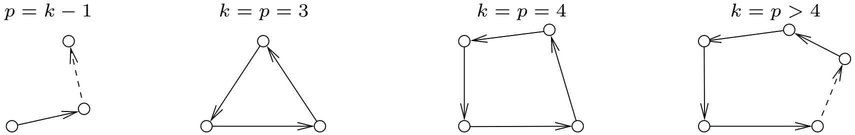
Finding The Translation Component in Eq. (29)

From (29) and (30): $0 < d \le 3$ – rank of camera center set, p – #pairs, k – #cameras

$$\hat{\mathbf{R}}_{ij}\mathbf{t}_i + s_{ij}\hat{\mathbf{t}}_{ij} - \mathbf{t}_j = \mathbf{0}, \qquad \sum_{i=1}^{k}\mathbf{t}_i = \mathbf{0}, \qquad \sum_{i,j} s_{ij} = p, \qquad s_{ij} > 0, \qquad \mathbf{t}_i \in \mathbb{R}^d$$

- in rank d: $d \cdot p + d + 1$ indep. eqns for $d \cdot k + p$ unknowns $\to p \ge \frac{d(k-1)-1}{d-1} \stackrel{\text{def}}{=} Q(d,k)$

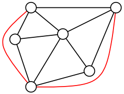
Ex: Chains and circuits construction from sticks of known orientation and unknown length?



$k \le 2$ for any d | $3 \ge d \ge 2$: non-collinear ok | $3 \ge d \ge 3$: non-planar ok | $3 \ge d \ge k - 1$: impossible

- equations insufficient for chains, trees, or when $d = 1$ — collinear cameras
- 3-connectivity implies sufficient equations for $d = 3$ — cams. in general pos. in 3D
  - s-connected graph has $p \ge \lceil \frac{sk}{2} \rceil$ edges for $s \ge 2$, hence $p \ge \lceil \frac{3k}{2} \rceil \ge Q(3,k) = \frac{3k}{2} - 2$
- 4-connectivity implies sufficient eqns. for any k when $d = 2$ — coplanar cams
  - since $p \ge \lceil 2k \rceil \ge Q(2,k) = 2k - 3$
  - maximal planar tringulated graphs have $p = 3k - 6$ and give a solution for $k \ge 3$

maximal planar triangulated graph example:



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Linear equations in (29) and (30) can be rewritten to

$$\mathbf{D}\mathbf{t} = \mathbf{0}, \qquad \mathbf{t} = \left[\mathbf{t}_1^\top, \mathbf{t}_2^\top, \ldots, \mathbf{t}_k^\top, s_{12}, \ldots, s_{ij}, \ldots\right]^\top$$

assuming measurement errors $\mathbf{D}\mathbf{t} = \boldsymbol{\epsilon}$ and $d = 3$, we have

$$\mathbf{t} \in \mathbb{R}^{3k+p}, \quad \mathbf{D} \in \mathbb{R}^{3p,3k+p} \quad \text{sparse}$$

and

$$\mathbf{t}^* = \arg\min_{\mathbf{t},\, s_{ij}>0} \mathbf{t}^\top \mathbf{D}^\top \mathbf{D}\, \mathbf{t}$$

- this is a quadratic programming problem (mind the constraints!)

```
z = zeros(3*k+p,1);
t = quadprog(D.'*D, z, diag([zeros(3*k,1); -ones(p,1)]), z);
```

- but check the rank first!

▶Bundle Adjustment

Goal: Use a good (and expensive) error model and improve all estimated parameters

Given:

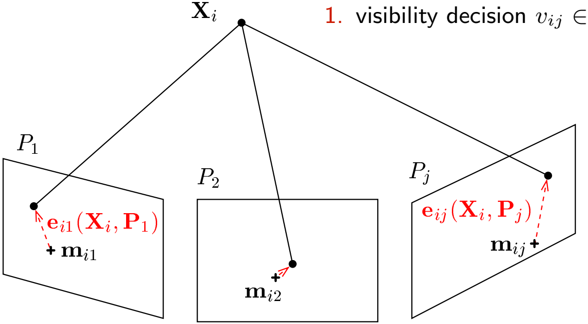
1. set of 3D points $\{\mathbf{X}_i\}_{i=1}^{p}$
2. set of cameras $\{\mathbf{P}_j\}_{j=1}^{c}$
3. fixed tentative projections $\mathbf{m}_{ij}$

Required:

1. corrected 3D points $\{\mathbf{X}'_i\}_{i=1}^{p}$
2. corrected cameras $\{\mathbf{P}'_j\}_{j=1}^{c}$

Latent:

1. visibility decision $v_{ij} \in \{0, 1\}$ per $\mathbf{m}_{ij}$



- for simplicity, $\mathbf{X}$, $\mathbf{m}$ are considered Cartesian (not homogeneous)
- we have projection error $\mathbf{e}_{ij}(\mathbf{X}_i, \mathbf{P}_j) = \mathbf{x}_i - \mathbf{m}_i$ per image feature, where $\underline{\mathbf{x}}_i = \mathbf{P}_j \underline{\mathbf{X}}_i$
- for simplicity, we will work with scalar error $e_{ij} = \|\mathbf{e}_{ij}\|$

Robust Objective Function for Bundle Adjustment

The data model is constructed by marginalization over v_{ij}, as in the Robust Matching Model →114

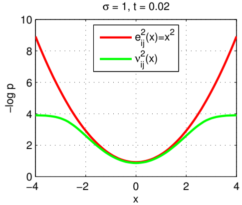
$$p(\{\mathbf{e}\} \mid \{\mathbf{P},\mathbf{X}\}) = \prod_{\text{pts}:i=1}^{p} \prod_{\text{cams}:j=1}^{c} \Big((1-P_0)p_1(e_{ij} \mid \mathbf{X}_i,\mathbf{P}_j) + P_0\, p_0(e_{ij} \mid \mathbf{X}_i,\mathbf{P}_j) \Big)$$

marginalized negative log-density is (→115)

$$-\log p(\{\mathbf{e}\} \mid \{\mathbf{P},\mathbf{X}\}) = \sum_i \sum_j \underbrace{-\log\Big(e^{-\frac{e_{ij}^2(\mathbf{X}_i,\mathbf{P}_j)}{2\sigma_1^2}} + t\Big)}_{\rho(e_{ij}^2(\mathbf{X}_i,\mathbf{P}_j)) = \nu_{ij}^2(\mathbf{X}_i,\mathbf{P}_j)} \stackrel{\text{def}}{=} \sum_i \sum_j \nu_{ij}^2(\mathbf{X}_i,\mathbf{P}_j)$$

- we can use LM, e_{ij} is the projection error (not Sampson error)
- ν_{ij} is a 'robust' error fcn.; it is non-robust ($\nu_{ij} = e_{ij}$) when $t = 0$
- $\rho(\cdot)$ is a 'robustification function' often found in M-estimation
- the $\mathbf{L}_{ij}$ in Levenberg-Marquardt changes to vector

$$(\mathbf{L}_{ij})_l = \frac{\partial \nu_{ij}}{\partial \theta_l} = \underbrace{\frac{1}{1 + t\, e^{e_{ij}^2(\theta)/(2\sigma_1^2)}}}_{\text{small for } e_{ij} \gg \sigma_1} \cdot \frac{1}{\nu_{ij}(\theta)} \cdot \frac{1}{4\sigma_1^2} \cdot \frac{\partial e_{ij}^2(\theta)}{\partial \theta_l} \qquad (32)$$



but the LM method stays the same as before →108–109

- outliers (wrong v_{ij}): almost no impact on $\mathbf{d}_s$ in normal equations because the red term in (32) scales contributions to both sums down for the particular ij

$$-\sum_{i,j} \mathbf{L}_{ij}^\top \nu_{ij}(\theta^s) = \Big(\sum_{i,j}^{k} \mathbf{L}_{ij}^\top \mathbf{L}_{ij}\Big)\mathbf{d}_s$$

▶Sparsity in Bundle Adjustment

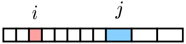
We have $q = 3p + 11k$ parameters: $\theta = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_p; \mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_k)$ points, cameras

We will use a multi-index $r = 1, \ldots, z$, $z = p \cdot k$. Then r correspond to point-cam pairs (i, j)

$$\theta^* = \arg\min_{\theta} \sum_{r=1}^{z} \nu_r^2(\theta), \quad \boldsymbol{\theta}^{s+1} := \boldsymbol{\theta}^s + \mathbf{d}_s, \quad -\sum_{r=1}^{z} \mathbf{L}_r^\top \nu_r(\theta^s) = \left(\sum_{r=1}^{z} \mathbf{L}_r^\top \mathbf{L}_r + \lambda \operatorname{diag}(\mathbf{L}_r^\top \mathbf{L}_r) \right) \mathbf{d}_s$$

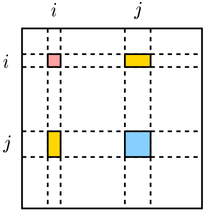
The block-form of $\mathbf{L}_r$ in Levenberg-Marquardt (→108) is zero except in columns i and j:

r-th error term is $\nu_r^2 = \rho(e_{ij}^2(\mathbf{X}_i, \mathbf{P}_j))$

$\mathbf{L}_r =$ 

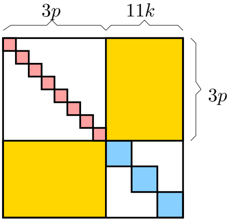
$r = (i, j)$ blocks:

- ■: $\mathbf{X}_i$, 1×3
- ■: $\mathbf{P}_j$, 1×11

$\mathbf{L}_r^\top \mathbf{L}_r =$ 

blocks:

- ■: $\mathbf{X}_i - \mathbf{X}_i$, 3×3
- ■: $\mathbf{X}_i - \mathbf{P}_j$, 3×11
- ■: $\mathbf{P}_j - \mathbf{P}_j$, 11×11

$\sum_{r=1}^{z} \mathbf{L}_r^\top \mathbf{L}_r =$ 

- "points-first-then-cameras" parameterization scheme

Thank You