►Summary: Projection Matrix of a General Finite Perspective Camera

$$\underline{\mathbf{m}} \simeq \mathbf{P}\underline{\mathbf{X}}, \qquad \mathbf{P} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \simeq \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \mathbf{K}\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix}$$

a recipe for filling $\mathbf{P}$

general finite perspective camera has 11 parameters:

- 5 intrinsic parameters: f, u_0, v_0, a, θ
- 6 extrinsic parameters: $\mathbf{t}$, $\mathbf{R}(\alpha, \beta, \gamma)$

finite camera: $\det \mathbf{K} \neq 0$

Representation Theorem: The set of projection matrices $\mathbf{P}$ of finite perspective cameras is isomorphic to the set of homogeneous 3×4 matrices with the left 3×3 submatrix $\mathbf{Q}$ non-singular.

random finite camera: `Q = rand(3,3); while det(Q)==0, Q = rand(3,3); end, P = [Q, rand(3,1)];`

▶Projection Matrix Decomposition

$$\mathbf{P} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \longrightarrow \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}$$

$\mathbf{Q} \in \mathbb{R}^{3,3}$ full rank (if finite perspective camera; see [H&Z, Sec. 6.3] for cameras at infinity)
$\mathbf{K} \in \mathbb{R}^{3,3}$ upper triangular with positive diagonal elements
$\mathbf{R} \in \mathbb{R}^{3,3}$ rotation mtx: $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$ and $\det \mathbf{R} = +1$

1. $\begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \begin{bmatrix}\mathbf{KR} & \mathbf{Kt}\end{bmatrix}$ also →35
2. RQ decomposition of $\mathbf{Q} = \mathbf{KR}$ using three Givens rotations [H&Z, p. 579]

$$\mathbf{K} = \mathbf{Q}\underbrace{\mathbf{R}_{32}\mathbf{R}_{31}\mathbf{R}_{21}}_{\mathbf{R}^{-1}} \qquad \mathbf{Q}\mathbf{R}_{32} = \begin{bmatrix}\cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\\ \cdot & 0 & \cdot\end{bmatrix}, \; \mathbf{Q}\mathbf{R}_{32}\mathbf{R}_{31} = \begin{bmatrix}\cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\\ 0 & 0 & \cdot\end{bmatrix}, \; \mathbf{Q}\mathbf{R}_{32}\mathbf{R}_{31}\mathbf{R}_{21} = \begin{bmatrix}\cdot & \cdot & \cdot\\ 0 & \cdot & \cdot\\ 0 & 0 & \cdot\end{bmatrix}$$

$\mathbf{R}_{ij}$ zeroes element ij in $\mathbf{Q}$ affecting only columns i and j and the sequence preserves previously zeroed elements, e.g. (see next slide for derivation details)

$$\mathbf{R}_{32} = \begin{bmatrix}1 & 0 & 0\\ 0 & c & -s\\ 0 & s & c\end{bmatrix} \text{ gives } \begin{array}{c} c^2 + s^2 = 1 \\ 0 = k_{32} = c\,q_{32} + s\,q_{33}\end{array} \Rightarrow c = \frac{q_{33}}{\sqrt{q_{32}^2 + q_{33}^2}} \quad s = \frac{-q_{32}}{\sqrt{q_{32}^2 + q_{33}^2}}$$

⊛ P1; 1pt: Multiply known matrices $\mathbf{K}$, $\mathbf{R}$ and then decompose back; discuss numerical errors

- RQ decomposition nonuniqueness: $\mathbf{KR} = \mathbf{KT}^{-1}\mathbf{TR}$, where $\mathbf{T} = \operatorname{diag}(-1,-1,1)$ is also a rotation, we must correct the result so that the diagonal elements of $\mathbf{K}$ are all positive
  'thin' RQ decomposition
- care must be taken to avoid overflow, see [Golub & van Loan 2013, sec. 5.2]

RQ Decomposition Step

```
Q = Array [q#1,#2 &, {3, 3}];
R32 = {{1, 0, 0}, {0, c, -s}, {0, s, c}}; R32 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}$$

```
Q1 = Q.R32 ; Q1 // MatrixForm
```

$$\begin{pmatrix} q_{1,1} & c\, q_{1,2} + s\, q_{1,3} & -s\, q_{1,2} + c\, q_{1,3} \\ q_{2,1} & c\, q_{2,2} + s\, q_{2,3} & -s\, q_{2,2} + c\, q_{2,3} \\ q_{3,1} & c\, q_{3,2} + s\, q_{3,3} & -s\, q_{3,2} + c\, q_{3,3} \end{pmatrix}$$

```
s1 = Solve [{Q1[[3]][[2]] == 0, c^2 + s^2 == 1}, {c, s}][[2]]
```

$$\left\{ c \to \frac{q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}},\ s \to -\frac{q_{3,2}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} \right\}$$

```
Q1 /. s1 // Simplify // MatrixForm
```

$$\begin{pmatrix} q_{1,1} & \frac{-q_{1,3}\, q_{3,2} + q_{1,2}\, q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} & \frac{q_{1,2}\, q_{3,2} + q_{1,3}\, q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} \\ q_{2,1} & \frac{-q_{2,3}\, q_{3,2} + q_{2,2}\, q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} & \frac{q_{2,2}\, q_{3,2} + q_{2,3}\, q_{3,3}}{\sqrt{q_{3,2}^2 + q_{3,3}^2}} \\ q_{3,1} & 0 & \sqrt{q_{3,2}^2 + q_{3,3}^2} \end{pmatrix}$$

▶Center of Projection (Optical Center)

Observation: finite $\mathbf{P}$ has a non-trivial right null-space — rank 3 but 4 columns

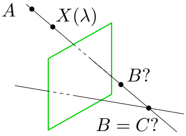
Theorem

Let $\mathbf{P}$ be a camera and let there be $\underline{\mathbf{B}} \neq \mathbf{0}$ s.t. $\mathbf{P}\,\underline{\mathbf{B}} = \mathbf{0}$. Then $\underline{\mathbf{B}}$ is equivalent to the projection center $\underline{\mathbf{C}}$ (homogeneous, in world coordinate frame).

Proof.

1. Let AB be a spatial line (B given from $\mathbf{P}\underline{\mathbf{B}} = \mathbf{0}$, $A \neq B$). Then

$$\underline{\mathbf{X}}(\lambda) \simeq \lambda\,\underline{\mathbf{A}} + (1-\lambda)\,\underline{\mathbf{B}}, \qquad \lambda \in \mathbb{R}$$



2. It projects to

$$\mathbf{P}\underline{\mathbf{X}}(\lambda) \simeq \lambda\,\mathbf{P}\,\underline{\mathbf{A}} + (1-\lambda)\,\mathbf{P}\,\underline{\mathbf{B}} \simeq \mathbf{P}\,\underline{\mathbf{A}}$$

- the entire line projects to a single point $\Rightarrow$ it must pass through the projection center of $\mathbf{P}$
- this holds for any choice of $A \neq B$ $\Rightarrow$ the only common point of the lines is the C, i.e. $\underline{\mathbf{B}} \simeq \underline{\mathbf{C}}$

□

Hence

$$\mathbf{0} = \mathbf{P}\,\underline{\mathbf{C}} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix}\begin{bmatrix}\mathbf{C} \\ 1\end{bmatrix} = \mathbf{Q}\,\mathbf{C} + \mathbf{q} \;\Rightarrow\; \mathbf{C} = -\mathbf{Q}^{-1}\mathbf{q}$$

$\underline{\mathbf{C}} = (c_j)$, where $c_j = (-1)^j \det \mathbf{P}^{(j)}$, in which $\mathbf{P}^{(j)}$ is $\mathbf{P}$ with column j dropped

Matlab: `C_homo = null(P);` or `C = -Q\q;`

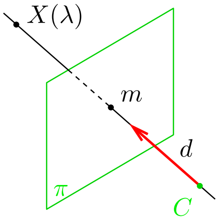
▶Optical Ray

Optical ray: Spatial line that projects to a single image point.

1. Consider the following spatial line
   $\mathbf{d} \in \mathbb{R}^3$ line direction vector, $\|\mathbf{d}\| = 1$, $\lambda \in \mathbb{R}$, Cartesian representation

$$\mathbf{X}(\lambda) = \mathbf{C} + \lambda\, \mathbf{d}$$

2. The projection of the (finite) point $X(\lambda)$ is



$$\underline{\mathbf{m}} \simeq \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \begin{bmatrix}\mathbf{X}(\lambda) \\ 1\end{bmatrix} = \mathbf{Q}(\mathbf{C} + \lambda \mathbf{d}) + \mathbf{q} = \lambda\, \mathbf{Q}\, \mathbf{d} =$$

$$= \lambda \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \begin{bmatrix}\mathbf{d} \\ 0\end{bmatrix}$$

...which is also the image of a point at infinity in $\mathbb{P}^3$

- optical ray line corresponding to image point m is the set

$$\mathbf{X}(\mu) = \mathbf{C} + \mu\, \mathbf{Q}^{-1} \underline{\mathbf{m}}, \qquad \mu \in \mathbb{R} \qquad (\mu = 1/\lambda)$$

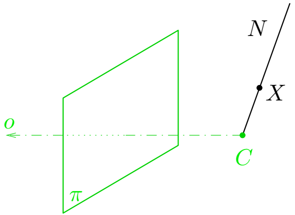
- optical ray direction may be represented by a point at infinity $(\mathbf{d}, 0)$ in $\mathbb{P}^3$
- optical ray is expressed in world coordinate frame

▶Optical Axis

Optical axis: Optical ray that is perpendicular to image plane π

1. points X on a given line N parallel to π project to a point at infinity $(u, v, 0)$ in π:

$$\begin{bmatrix} u \\ v \\ 0 \end{bmatrix} \simeq \mathbf{P}\underline{\mathbf{X}} = \begin{bmatrix} \mathbf{q}_1^\top & q_{14} \\ \mathbf{q}_2^\top & q_{24} \\ \mathbf{q}_3^\top & q_{34} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$



2. therefore the set of points X is parallel to π iff

$$\mathbf{q}_3^\top \mathbf{X} + q_{34} = 0$$

3. this is a plane with $\pm\mathbf{q}_3$ as the normal vector
4. optical axis direction: substitution $\mathbf{P} \mapsto \lambda\mathbf{P}$ must not change the direction
5. we select (assuming $\det(\mathbf{R}) > 0$)

$$\mathbf{o} = \det(\mathbf{Q})\, \mathbf{q}_3$$

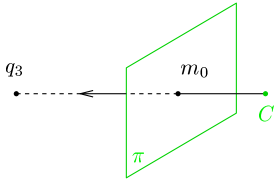
if $\mathbf{P} \mapsto \lambda\mathbf{P}$ then $\det(\mathbf{Q}) \mapsto \lambda^3 \det(\mathbf{Q})$ and $\mathbf{q}_3 \mapsto \lambda\, \mathbf{q}_3$ [H&Z, p. 161]

- the axis is expressed in world coordinate frame

▶Principal Point

Principal point: The intersection of image plane and the optical axis

1. as we saw, $\mathbf{q}_3$ is the directional vector of optical axis
2. we take point at infinity on the optical axis that must project to the principal point m_0



3. then

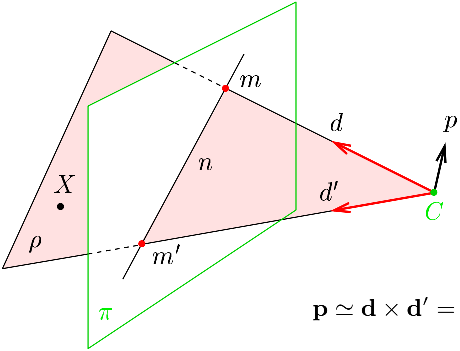
$$\underline{\mathbf{m}}_0 \simeq \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} \begin{bmatrix}\mathbf{q}_3 \\ 0\end{bmatrix} = \mathbf{Q}\,\mathbf{q}_3$$

principal point: $\underline{\mathbf{m}}_0 \simeq \mathbf{Q}\,\mathbf{q}_3$

- principal point is also the center of radial distortion

▶Optical Plane

A spatial plane with normal p containing the projection center C and a given image line n.



optical ray given by m $\quad \mathbf{d} \simeq \mathbf{Q}^{-1}\underline{\mathbf{m}}$

optical ray given by m' $\quad \mathbf{d}' \simeq \mathbf{Q}^{-1}\underline{\mathbf{m}}'$

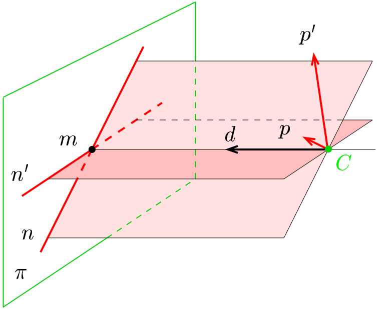
$$\mathbf{p} \simeq \mathbf{d} \times \mathbf{d}' = (\mathbf{Q}^{-1}\underline{\mathbf{m}}) \times (\mathbf{Q}^{-1}\underline{\mathbf{m}}') = \mathbf{Q}^\top(\underline{\mathbf{m}} \times \underline{\mathbf{m}}') = \mathbf{Q}^\top\underline{\mathbf{n}}$$

- note the way $\mathbf{Q}$ factors out!

hence, $0 = \mathbf{p}^\top(\mathbf{X} - \mathbf{C}) = \underline{\mathbf{n}}^\top \underbrace{\mathbf{Q}(\mathbf{X} - \mathbf{C})}_{\rightarrow 30} = \underline{\mathbf{n}}^\top\mathbf{P}\underline{\mathbf{X}} = (\mathbf{P}^\top\underline{\mathbf{n}})^\top\underline{\mathbf{X}}$ for every X in plane ρ

optical plane is given by n: $\quad \underline{\boldsymbol{\rho}} \simeq \mathbf{P}^\top\underline{\mathbf{n}}$ $\qquad \rho_1\,x + \rho_2\,y + \rho_3\,z + \rho_4 = 0$

Cross-Check: Optical Ray as Optical Plane Intersection



optical plane normal given by n $\qquad \mathbf{p} = \mathbf{Q}^\top \underline{\mathbf{n}}$

optical plane normal given by n' $\qquad \mathbf{p}' = \mathbf{Q}^\top \underline{\mathbf{n}'}$

$$\mathbf{d} = \mathbf{p} \times \mathbf{p}' = (\mathbf{Q}^\top \underline{\mathbf{n}}) \times (\mathbf{Q}^\top \underline{\mathbf{n}}') = \mathbf{Q}^{-1}(\underline{\mathbf{n}} \times \underline{\mathbf{n}}') = \mathbf{Q}^{-1}\underline{\underline{\mathbf{m}}}$$

►Summary: Projection Center; Optical Ray, Axis, Plane

General (finite) camera

$$\mathbf{P} = \begin{bmatrix}\mathbf{Q} & \mathbf{q}\end{bmatrix} = \begin{bmatrix}\mathbf{q}_1^\top & q_{14}\\ \mathbf{q}_2^\top & q_{24}\\ \mathbf{q}_3^\top & q_{34}\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix} = \mathbf{K}\,\mathbf{R}\begin{bmatrix}\mathbf{I} & -\mathbf{C}\end{bmatrix}$$

$\underline{\mathbf{C}} \simeq \operatorname{rnull}(\mathbf{P}), \quad \mathbf{C} = -\mathbf{Q}^{-1}\mathbf{q}$ — projection center (world coords.) →35

$\mathbf{d} = \mathbf{Q}^{-1}\,\underline{\mathbf{m}}$ — optical ray direction (world coords.) →36

$\mathbf{o} = \det(\mathbf{Q})\,\mathbf{q}_3$ — outward optical axis (world coords.) →37

$\underline{\mathbf{m}}_0 \simeq \mathbf{Q}\,\mathbf{q}_3$ — principal point (in image plane) →38

$\underline{\boldsymbol{\rho}} = \mathbf{P}^\top\,\underline{\mathbf{n}}$ — optical plane (world coords.) →39

$$\mathbf{K} = \begin{bmatrix}a\,f & -a\,f\,\cot\theta & u_0\\ 0 & f/\sin\theta & v_0\\ 0 & 0 & 1\end{bmatrix}$$
camera (calibration) matrix (f, u_0, v_0 in pixels) →31

$\mathbf{R}$ — camera rotation matrix (cam coords.) →30

$\mathbf{t}$ — camera translation vector (cam coords.) →30

What Can We Do with An 'Uncalibrated' Perspective Camera?

How far is the engine?

distance between sleepers (ties) 0.806m but we cannot count them, the image resolution is too low

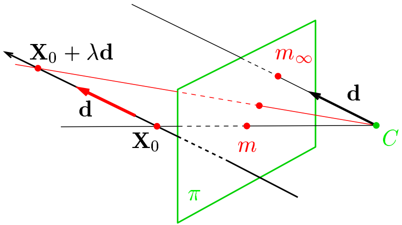
We will review some life-saving theory. . .
. . . and build a bit of geometric intuition. . .

In fact

- 'uncalibrated' = the image contains a 'calibrating object' that suffices for the task at hand

▶Vanishing Point

Vanishing point: the limit of the projection of a point that moves along a space line infinitely in one direction. the image of the point at infinity on the line



$$\underline{\mathbf{m}}_\infty \simeq \lim_{\lambda \to \pm\infty} \mathbf{P} \begin{bmatrix} \mathbf{X}_0 + \lambda\mathbf{d} \\ 1 \end{bmatrix} = \cdots \simeq \mathbf{Q}\,\mathbf{d}$$

⊛ P1; 1pt: Prove (use Cartesian coordinates and L'Hôpital's rule)

- the V.P. of a spatial line with directional vector $\mathbf{d}$ is $\underline{\mathbf{m}}_\infty \simeq \mathbf{Q}\,\mathbf{d}$
- V.P. is independent on line position $\mathbf{X}_0$, it depends on its directional vector only
- all parallel (world) lines share the same (image) V.P., including the optical ray defined by m_∞

Some Vanishing Point "Applications"

where is the sun?

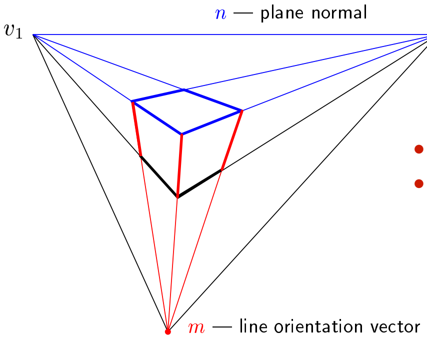
what is the wind direction?
(must have video)

fly above the lane,
at constant altitude!

▶Vanishing Line

Vanishing line: The set of vanishing points of all lines in a plane

the image of the line at infinity in the plane
and in all parallel planes (!)



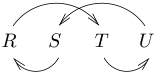
- any box with parallel edges
- top (blue) and bottom (black) box planes are parallel, hence they share V.L. n

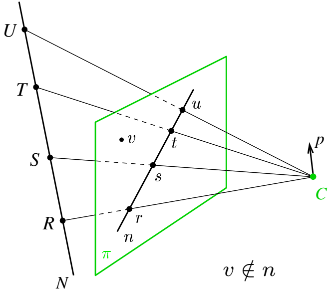
- V.L. n corresponds to spatial plane of normal vector $\mathbf{p} = \mathbf{Q}^\top \underline{\mathbf{n}}$
  because this is the normal vector of a parallel optical plane (!) →39
- a spatial plane of normal vector $\mathbf{p}$ has a V.L. represented by $\underline{\mathbf{n}} = \mathbf{Q}^{-\top} \mathbf{p}$.

▶Cross Ratio

Four distinct collinear spatial points R, S, T, U define cross-ratio

$$[RSTU] = \frac{|\overrightarrow{RT}|}{|\overrightarrow{SR}|} \frac{|\overrightarrow{US}|}{|\overrightarrow{TU}|}$$





$|\overrightarrow{RT}|$ – signed distance from R to T in the arrow direction

6 cross-ratios from four points:

$$[SRUT] = [RSTU], \ [RSUT] = \frac{1}{[RSTU]}, \ [RTSU] = 1 - [RSTU], \ \cdots$$

Obs: $$[RSTU] = \frac{\begin{vmatrix}\underline{\mathbf{r}} & \underline{\mathbf{t}} & \underline{\mathbf{v}}\end{vmatrix}}{\begin{vmatrix}\underline{\mathbf{s}} & \underline{\mathbf{r}} & \underline{\mathbf{v}}\end{vmatrix}} \cdot \frac{\begin{vmatrix}\underline{\mathbf{u}} & \underline{\mathbf{s}} & \underline{\mathbf{v}}\end{vmatrix}}{\begin{vmatrix}\underline{\mathbf{t}} & \underline{\mathbf{u}} & \underline{\mathbf{v}}\end{vmatrix}}, \quad \begin{vmatrix}\underline{\mathbf{r}} & \underline{\mathbf{t}} & \underline{\mathbf{v}}\end{vmatrix} = \det\begin{bmatrix}\underline{\mathbf{r}} & \underline{\mathbf{t}} & \underline{\mathbf{v}}\end{bmatrix} = (\underline{\mathbf{r}} \times \underline{\mathbf{t}})^\top \underline{\mathbf{v}} \qquad (1)$$

Corollaries:

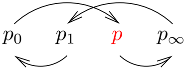
- cross ratio is invariant under homographies $\underline{\mathbf{x}}' \simeq \mathbf{H}\underline{\mathbf{x}}$ plug $\mathbf{H}\underline{\mathbf{x}}$ in (1): $(\mathbf{H}^{-\top}(\underline{\mathbf{r}} \times \underline{\mathbf{t}}))^\top \mathbf{H}\underline{\mathbf{v}}$
- cross ratio is invariant under perspective projection: $[RSTU] = [r\,s\,t\,u]$

- 4 collinear points: any perspective camera will "see" the same cross-ratio of their images
- we measure the same cross-ratio in image as on the world line
- one of the points R, S, T, U may be at infinity (we take the limit, in effect $\frac{\infty}{\infty} = 1$)

▶1D Projective Coordinates

The 1-D projective coordinate of a point P is defined by the following cross-ratio:

$$[P] = [P_0\, P_1\, P\, P_\infty] = [p_0\, p_1\, p\, p_\infty] = \frac{|\overrightarrow{p_0\, p}|}{|\overrightarrow{p_1\, p_0}|}\,\frac{|\overrightarrow{p_\infty\, p_1}|}{|\overrightarrow{p\, p_\infty}|} = [p]$$

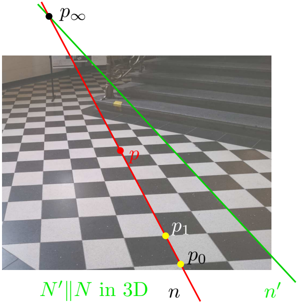


naming convention:

P_0 – the origin	$[P_0] = 0$
P_1 – the unit point	$[P_1] = 1$
P_∞ – the supporting point	$[P_\infty] = \pm\infty$

$$[P] = [p]$$

$[P]$ is equal to Euclidean coordinate along N
$[p]$ is its measurement in the image plane



Applications

- Given the image of a 3D line N, the origin, the unit point, and the vanishing point, then the Euclidean coordinate of any point $P \in N$ can be determined →48
- Finding v.p. of a line through a regular object →49

Thank You