3D Computer Vision

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Open Informatics Master's Course

►Choleski Decomposition for B. A.

The most expensive computation in B. A. is solving the normal eqs:

$$\text{find } \mathbf{x} \text{ such that} \quad -\sum_{r=1}^{z} \mathbf{L}_r^\top \nu_r(\theta^s) = \Big(\sum_{r=1}^{z} \mathbf{L}_r^\top \mathbf{L}_r + \lambda \operatorname{diag}\big(\mathbf{L}_r^\top \mathbf{L}_r\big)\Big)\mathbf{x}$$

- $\mathbf{A}$ is very large — approx. $3 \cdot 10^4 \times 3 \cdot 10^4$ for a small problem of 10000 points and 5 cameras
- $\mathbf{A}$ is sparse and symmetric, $\mathbf{A}^{-1}$ is dense — direct matrix inversion is prohibitive

> Choleski: symmetric positive definite matrix $\mathbf{A}$ can be decomposed to $\mathbf{A} = \mathbf{L}\mathbf{L}^\top$, where $\mathbf{L}$ is lower triangular. If $\mathbf{A}$ is sparse then $\mathbf{L}$ is sparse, too.

1. decompose $\mathbf{A} = \mathbf{L}\mathbf{L}^\top$ — transforms the problem to $\mathbf{L}\underbrace{\mathbf{L}^\top\mathbf{x}}_{\mathbf{c}} = \mathbf{b}$
2. solve for $\mathbf{x}$ in two passes:

$$\mathbf{L}\,\mathbf{c} = \mathbf{b} \qquad \mathbf{c}_i := \mathbf{L}_{ii}^{-1}\Big(\mathbf{b}_i - \sum_{j<i} \mathbf{L}_{ij}\mathbf{c}_j\Big)$$

forward substitution, $i = 1, \dots, q$ (params)

$$\mathbf{L}^\top\mathbf{x} = \mathbf{c} \qquad \mathbf{x}_i := \mathbf{L}_{ii}^{-1}\Big(\mathbf{c}_i - \sum_{j>i} \mathbf{L}_{ji}\mathbf{x}_j\Big)$$

back-substitution

- Choleski decomposition is fast (does not touch zero blocks)
  non-zero elements are $9p + 121k + 66pk \approx 3.4 \cdot 10^6$; ca. $250\times$ fewer than all elements
- it can be computed on single elements or on entire blocks
- use profile Choleski for sparse $\mathbf{A}$ and diagonal pivoting for semi-definite $\mathbf{A}$ — see above; [Triggs et al. 1999]
- λ controls the definiteness

Profile Choleski Decomposition is Simple

```
function L = pchol(A)
%
% PCHOL profile Choleski factorization,
%    L = PCHOL(A) returns lower-triangular sparse L such that A = L*L'
%    for sparse square symmetric positive definite matrix A,
%    especially efficient for arrowhead sparse matrices.

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 [p,q] = size(A);
 if p ~= q, error 'Matrix A is not square'; end

 L = sparse(q,q);
 F = ones(q,1);
 for i=1:q
  F(i) = find(A(i,:),1); % 1st non-zero on row i; we are building F gradually
  for j = F(i):i-1
   k = max(F(i),F(j));
   a = A(i,j) - L(i,k:(j-1))*L(j,k:(j-1))';
   L(i,j) = a/L(j,j);
  end
  a = A(i,i) - sum(full(L(i,F(i):(i-1))).^2);
  if a < 0, error 'Matrix A is not positive definite'; end
  L(i,i) = sqrt(a);
 end
end
```

►Gauge Freedom (kalibrační invariance)

1. The external frame is not fixed: See Projective Reconstruction Theorem →132

$$\underline{\mathbf{m}}_{ij} \simeq \mathbf{P}_j \underline{\mathbf{X}}_i = \mathbf{P}_j \mathbf{H}^{-1} \mathbf{H} \underline{\mathbf{X}}_i = \mathbf{P}'_j \underline{\mathbf{X}}'_i$$

2. Some representations are not minimal, e.g.
   - $\mathbf{P}$ is 12 numbers for 11 parameters
   - we may represent $\mathbf{P}$ in decomposed form $\mathbf{K}$, $\mathbf{R}$, $\mathbf{t}$
   - but $\mathbf{R}$ is 9 numbers representing the 3 parameters of rotation

As a result

- there is no unique solution
- matrix $\sum_r \mathbf{L}_r^\top \mathbf{L}_r$ is singular

Solutions

1. fixing the external frame (e.g. a selected camera frame) explicitly or by constraints
2. fixing the scale (e.g. $s_{12} = 1$)

3a. either imposing constraints on projective entities
   - cameras, e.g. $\mathbf{P}_{3,4} = 1$ — this excludes affine cameras
   - points, e.g. $(\underline{\mathbf{X}}_i)_4 = 1$ or $\|\underline{\mathbf{X}}_i\|^2 = 1$ — the 2nd: can represent points at infinity

3b. or using minimal representations
   - points in their Euclidean representation $\mathbf{X}_i$ — but finite points may be an unrealistic model
   - rotation matrices can be represented by skew-symmetric matrices →149

Implementing Simple Linear Constraints (by programmatic elimination)

What for?

1. fixing external frame as in $\theta_i = \mathbf{t}_i$, $s_{kl} = 1$ for some i, k, l 'trivial gauge'
2. representing additional knowledge as in $\theta_i = \theta_j$ e.g. cameras share calibration matrix $\mathbf{K}$

Introduce reduced parameters $\hat{\theta}$ and replication matrix $\mathbf{T}$:

$$\theta = \mathbf{T}\,\hat{\theta} + \mathbf{t}, \quad \mathbf{T} \in \mathbb{R}^{p,\hat{p}}, \quad \hat{p} \le p$$

then $\mathbf{L}_r$ in LM changes to $\mathbf{L}_r\,\mathbf{T}$ and everything else stays the same →108

$\mathbf{T} =$

	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{\theta}_4$
θ_1	1			
θ_2		1		
θ_3				
θ_4				1
θ_5				1

$\mathbf{t} =$

1

these $\mathbf{T}$, $\mathbf{t}$ represent	
$\theta_1 = \hat{\theta}_1$	no change
$\theta_2 = \hat{\theta}_2$	no change
$\theta_3 = t_3$	constancy
$\theta_4 = \theta_5 = \hat{\theta}_4$	equality

- $\mathbf{T}$ deletes columns of $\mathbf{L}_r$ that correspond to fixed parameters it reduces the problem size
- consistent initialisation: $\theta^0 = \mathbf{T}\,\hat{\theta}^0 + \mathbf{t}$ or filter the init by pseudoinverse $\theta^0 \mapsto \mathbf{T}^\dagger \theta^0$
- no need for computing derivatives for θ_j corresponding to all-zero rows of $\mathbf{T}$ fixed θ

- constraining projective entities →149–151
- more complex constraints tend to make normal equations dense
- implementing constraints is safer than explicit renaming of the parameters, gives a flexibility to experiment
- other methods are much more involved, see [Triggs et al. 1999]
- **BA resource:** `http://www.ics.forth.gr/~lourakis/sba/` [Lourakis 2009]

Matrix Exponential: A path to Minimal Parameterizations

- for any square matrix we define

$$\operatorname{expm}(\mathbf{A}) = \sum_{k=0}^{\infty} \frac{1}{k!}\,\mathbf{A}^k \qquad \text{note: } \mathbf{A}^0 = \mathbf{I}$$

- some properties:

$$\operatorname{expm}(x) = e^x, \quad x \in \mathbb{R}, \quad \operatorname{expm}\mathbf{0} = \mathbf{I}, \quad \operatorname{expm}(-\mathbf{A}) = \big(\operatorname{expm}\mathbf{A}\big)^{-1},$$

$$\operatorname{expm}(a\mathbf{A} + b\mathbf{A}) = \operatorname{expm}(a\,\mathbf{A})\operatorname{expm}(b\,\mathbf{A}), \quad \operatorname{expm}(\mathbf{A}+\mathbf{B}) \neq \operatorname{expm}(\mathbf{A})\operatorname{expm}(\mathbf{B})$$

$\operatorname{expm}(\mathbf{A}^\top) = (\operatorname{expm}\mathbf{A})^\top$ hence if $\mathbf{A}$ is skew symmetric then $\operatorname{expm}\mathbf{A}$ is orthogonal:

$$\big(\operatorname{expm}(\mathbf{A})\big)^\top = \operatorname{expm}(\mathbf{A}^\top) = \operatorname{expm}(-\mathbf{A}) = \big(\operatorname{expm}(\mathbf{A})\big)^{-1}$$

$$\det\big(\operatorname{expm}\mathbf{A}\big) = e^{\operatorname{tr}\mathbf{A}}$$

Some consequences

- traceless matrices ($\operatorname{tr}\mathbf{A} = 0$) map to unit-determinant matrices $\Rightarrow$ we can represent homogeneous matrices
- skew-symmetric matrices map to orthogonal matrices $\Rightarrow$ we can represent rotations
- matrix exponential provides the exponential map from the powerful Lie group theory

Lie Groups Useful in 3D Vision

group		matrix	represent
special linear	$\mathrm{SL}(3,\mathbb{R})$	real 3×3, unit determinant $\mathbf{H}$	2D homography
special linear	$\mathrm{SL}(4,\mathbb{R})$	real 4×4, unit determinant	3D homography
special orthogonal	$\mathrm{SO}(3)$	real 3×3 orthogonal $\mathbf{R}$	3D rotation
special Euclidean	$\mathrm{SE}(3)$	4×4 $\begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$, $\mathbf{R} \in \mathrm{SO}(3)$, $\mathbf{t} \in \mathbb{R}^3$	3D rigid motion
similarity	$\mathrm{Sim}(3)$	4×4 $\begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & s^{-1} \end{bmatrix}$, $s \in \mathbb{R} \setminus 0$	rigid motion + scale

- Lie group G = topological group that is also a smooth manifold with nice properties
- Lie algebra $\mathfrak{g}$ = vector space associated with a Lie group (tangent space of the manifold)
- group: this is where we need to work
- algebra: this is how to represent group elements with a minimal number of parameters
- Exponential map = map between algebra and its group $\exp\colon \mathfrak{g} \to G$
- for matrices $\exp = \mathrm{expm}$
- in most of the above groups we a have a closed-form formula for the exponential and for its principal inverse
- Jacobians are also readily available for $\mathrm{SO}(3)$, $\mathrm{SE}(3)$ [Solà 2020]

Homography

$$\mathbf{H} = \operatorname{expm} \mathbf{Z}$$

- $\mathrm{SL}(3, \mathbb{R})$ group element

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \quad \text{s.t.} \quad \det \mathbf{H} = 1$$

- $\mathfrak{sl}(3, \mathbb{R})$ algebra element — 8 parameters

$$\mathbf{Z} = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & -(z_{11} + z_{22}) \end{bmatrix}$$

- note that $\operatorname{tr} \mathbf{Z} = 0$

▶Rotation in 3D

$$\mathbf{R} = \operatorname{expm}\,[\boldsymbol{\phi}]_\times, \quad \boldsymbol{\phi} = (\phi_1,\, \phi_2,\, \phi_3) = \varphi\,\mathbf{e}_\varphi, \quad 0 \leq \varphi < \pi, \quad \|\mathbf{e}_\varphi\| = 1$$

- $\mathrm{SO}(3)$ group element

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \quad \text{s.t.} \quad \mathbf{R}^{-1} = \mathbf{R}^\top$$

- $\mathfrak{so}(3)$ algebra element — 3 parameters

$$[\boldsymbol{\phi}]_\times = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix}$$

- exponential map in closed form — Rodrigues' formula

$$\mathbf{R} = \operatorname{expm}\,[\boldsymbol{\phi}]_\times = \sum_{n=0}^{\infty} \frac{[\boldsymbol{\phi}]_\times^n}{n!} = \overset{\circledast\,1}{\cdots} = \mathbf{I} + \frac{\sin\varphi}{\varphi}\,[\boldsymbol{\phi}]_\times + \frac{1-\cos\varphi}{\varphi^2}\,[\boldsymbol{\phi}]_\times^2$$

- (principal) logarithm — log is a periodic function

$$0 \leq \varphi < \pi, \quad \cos\varphi = \frac{1}{2}\left(\operatorname{tr}(\mathbf{R}) - 1\right), \quad [\boldsymbol{\phi}]_\times = \frac{\varphi}{2\,\sin\varphi}(\mathbf{R} - \mathbf{R}^\top),$$

- $\boldsymbol{\phi}$ is rotation axis vector $\mathbf{e}_\varphi$ scaled by rotation angle φ in radians
- finite limits for $\varphi \to 0$ exist: $\sin(\varphi)/\varphi \to 1$, $(1-\cos\varphi)/\varphi^2 \to 1/2$

3D Rigid Motion

$$\mathbf{M} = \operatorname{expm} [\boldsymbol{\nu}]_\wedge$$

- $\mathrm{SE}(3)$ group element — 4×4 matrix

$$\mathbf{M} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \quad \text{s.t.} \quad \mathbf{R} \in \mathrm{SO}(3),\ \mathbf{t} \in \mathbb{R}^3$$

- $\mathfrak{se}(3)$ algebra element — 4×4 matrix

$$[\boldsymbol{\nu}]_\wedge = \begin{bmatrix} [\boldsymbol{\phi}]_\times & \boldsymbol{\rho} \\ \mathbf{0} & 0 \end{bmatrix} \quad \text{s.t.} \quad \boldsymbol{\phi} \in \mathbb{R}^3,\ \varphi = \|\boldsymbol{\phi}\| < \pi,\ \boldsymbol{\rho} \in \mathbb{R}^3$$

- exponential map in closed form

$$\mathbf{R} = \operatorname{expm} [\boldsymbol{\phi}]_\times, \quad \mathbf{t} = \operatorname{dexpm}([\boldsymbol{\phi}]_\times)\, \boldsymbol{\rho}$$

$$\operatorname{dexpm}([\boldsymbol{\phi}]_\times) = \sum_{n=0}^{\infty} \frac{[\boldsymbol{\phi}]_\times^n}{(n+1)!} = \mathbf{I} + \frac{1 - \cos\varphi}{\varphi^2} [\boldsymbol{\phi}]_\times + \frac{\varphi - \sin\varphi}{\varphi^3} [\boldsymbol{\phi}]_\times^2$$

$$\operatorname{dexpm}^{-1}([\boldsymbol{\phi}]_\times) = \mathbf{I} - \frac{1}{2} [\boldsymbol{\phi}]_\times + \frac{1}{\varphi^2} \left(1 - \frac{\varphi}{2} \cot \frac{\varphi}{2} \right) [\boldsymbol{\phi}]_\times^2$$

- dexpm: differential of the exponential in $\mathrm{SO}(3)$
- (principal) logarithm via a similar trick as in $\mathrm{SO}(3)$
- finite limits exist: $(\varphi - \sin\varphi)/\varphi^3 \to 1/6$
- this form is preferred to $\mathrm{SO}(3) \times \mathbb{R}^3$

▶Minimal Representations for Other Entities

- fundamental matrix via $\mathrm{SO}(3) \times \mathrm{SO}(3) \times \mathbb{R}$

$$\mathbf{F} = \mathbf{U}\mathbf{D}\mathbf{V}^\top, \quad \mathbf{D} = \mathrm{diag}(1, d^2, 0), \quad \mathbf{U}, \mathbf{V} \in \mathrm{SO}(3), \quad 3 + 1 + 3 = 7 \text{ DOF}$$

- essential matrix via $\mathrm{SO}(3) \times \mathbb{R}^3$

$$\mathbf{E} = [-\mathbf{t}]_\times \mathbf{R}, \quad \mathbf{R} \in \mathrm{SO}(3), \quad \mathbf{t} \in \mathbb{R}^3, \ \|\mathbf{t}\| = 1, \qquad 3 + 2 = 5 \text{ DOF}$$

- camera pose via $\mathrm{SO}(3) \times \mathbb{R}^3$ or $\mathrm{SE}(3)$

$$\mathbf{P} = \mathbf{K} \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} = \begin{bmatrix} \mathbf{K} & \mathbf{0} \end{bmatrix} \mathbf{M}, \qquad 5 + 3 + 3 = 11 \text{ DOF}$$

- $\mathrm{Sim}(3)$ useful for SfM without scale
- closed-form formulae still exist but are a bit messy
- a (bit too brief) intro to Lie groups in 3D vision/robotics and SW:

J. Solà, J. Deray, and D. Atchuthan. A micro Lie theory for state estimation in robotics. arXiv:1812.01537v7 [cs.RO], August 2020.

Module VII

Stereovision

7.1 Introduction
7.2 Epipolar Rectification
7.3 Binocular Disparity and Matching Table
7.4 Image Similarity
7.5 Marroquin's Winner Take All Algorithm
7.6 Maximum Likelihood Matching
7.7 Uniqueness and Ordering as Occlusion Models

mostly covered by

Šára, R. How To Teach Stereoscopic Vision. Proc. ELMAR 2010 referenced as [SP]

additional references

C. Geyer and K. Daniilidis. Conformal rectification of omnidirectional stereo pairs. In *Proc Computer Vision and Pattern Recognition Workshop*, p. 73, 2003.

J. Gluckman and S. K. Nayar. Rectifying transformations that minimize resampling effects. In *Proc IEEE CS Conf on Computer Vision and Pattern Recognition*, vol. 1:111–117. 2001.

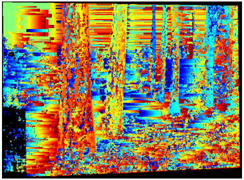
M. Pollefeys, R. Koch, and L. V. Gool. A simple and efficient rectification method for general motion. In *Proc Int Conf on Computer Vision*, vol. 1:496–501, 1999.

Stereovision: What Are The Relative Distances?

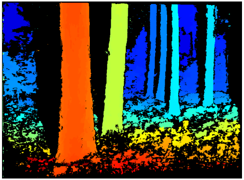
The success of a model-free stereo matching algorithm is unlikely:

WTA Matching:

for every left-image pixel find the most similar right-image pixel along the corresponding epipolar line [Marroquin 83]



disparity map from WTA



a good disparity map

- monocular vision already gives a rough 3D sketch because we understand the scene
- pixelwise independent matching without any understanding is difficult
- matching can benefit from a geometric simplification of the problem

▶Linear Epipolar Rectification for Easier Correspondence Search

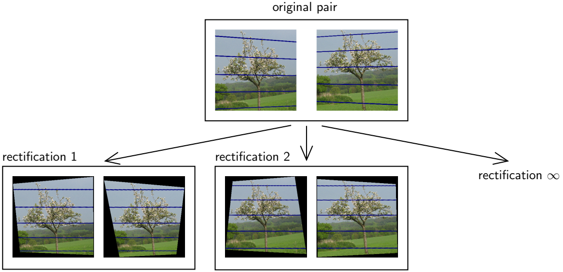
Obs:

- if we map epipoles to infinity, epipolar lines become parallel
- we then rotate them to become horizontal
- we then scale the images to make corresponding epipolar lines colinear
- this can be achieved by a pair of (non-unique) homographies applied to the images

Problem: Given fundamental matrix $\mathbf{F}$ or camera matrices $\mathbf{P}_1$, $\mathbf{P}_2$, compute a pair of homographies that maps epipolar lines to horizontal with the same row coordinate.

Procedure:

1. find a pair of rectification homographies $\mathbf{H}_1$ and $\mathbf{H}_2$.
2. warp images using $\mathbf{H}_1$ and $\mathbf{H}_2$ and transform the fundamental matrix $\mathbf{F} \mapsto \mathbf{H}_2^{-\top}\mathbf{F}\mathbf{H}_1^{-1}$ or the cameras $\mathbf{P}_1 \mapsto \mathbf{H}_1\mathbf{P}_1$, $\mathbf{P}_2 \mapsto \mathbf{H}_2\mathbf{P}_2$.

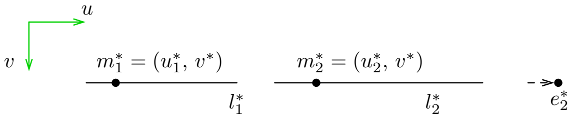


▶Rectification Homographies

Assumption: Cameras $(\mathbf{P}_1, \mathbf{P}_2)$ are rectified by a homography pair $(\mathbf{H}_1, \mathbf{H}_2)$:

$$\mathbf{P}_i^* = \mathbf{H}_i \mathbf{P}_i = \mathbf{H}_i \mathbf{K}_i \mathbf{R}_i \begin{bmatrix} \mathbf{I} & -\mathbf{C}_i \end{bmatrix}, \quad i = 1, 2$$

rectified entities: $\mathbf{F}^*$, l_1^*, l_2^*, etc:



- the rectified location difference $d = u_1^* - u_2^*$ is called disparity

corresponding epipolar lines must be:

1. parallel to image rows $\Rightarrow$ epipoles become $e_1^* = e_2^* = (1, 0, 0)$
2. equivalent $l_2^* = l_1^*$: $\quad \underline{\mathbf{l}}_1^* \simeq \underline{\mathbf{e}}_1^* \times \underline{\mathbf{m}}_1 = [\underline{\mathbf{e}}_1^*]_\times \underline{\mathbf{m}}_1 \simeq \underline{\mathbf{l}}_2^* \simeq \mathbf{F}^* \underline{\mathbf{m}}_1 \quad \Rightarrow \quad \mathbf{F}^* = [\underline{\mathbf{e}}_1^*]_\times$

- therefore the canonical fundamental matrix is

$$\mathbf{F}^* \simeq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

A two-step rectification procedure

1. find some pair of primitive rectification homographies $\hat{\mathbf{H}}_1$, $\hat{\mathbf{H}}_2$
2. upgrade to a pair of optimal rectification homographies while preserving $\mathbf{F}^*$

▶Geometric Interpretation of Linear Rectification

What pair of physical cameras is compatible with $\mathbf{F}^*$?

- we know that $\mathbf{F} = (\mathbf{Q}_1 \mathbf{Q}_2^{-1})^\top [\underline{\mathbf{e}}_1]_\times$ →79
- we choose $\mathbf{Q}_1^* = \mathbf{K}_1^*$, $\mathbf{Q}_2^* = \mathbf{K}_2^* \mathbf{R}^*$; then

$$\mathbf{F}^* \simeq (\mathbf{Q}_1^* \mathbf{Q}_2^{*\,-1})^\top [\underline{\mathbf{e}}_1^*]_\times \overset{!}{\simeq} (\mathbf{K}_1^* \mathbf{R}^{*\top} \mathbf{K}_2^{*\,-1})^\top \mathbf{F}^*$$

- we look for $\mathbf{R}^*$, $\mathbf{K}_1^*$, $\mathbf{K}_2^*$ compatible with

$$(\mathbf{K}_1^* \mathbf{R}^{*\top} \mathbf{K}_2^{*\,-1})^\top \mathbf{F}^* = \lambda \mathbf{F}^*, \qquad \mathbf{R}^* \mathbf{R}^{*\top} = \mathbf{I}, \qquad \mathbf{K}_1^*, \mathbf{K}_2^* \text{ upper triangular}$$

- we also want $\mathbf{b}^*$ from $\underline{\mathbf{e}}_1^* \simeq \mathbf{P}_1^* \underline{\mathbf{C}}_2^* = \mathbf{K}_1^* \mathbf{b}^*$ $\quad$ $\mathbf{b}^*$ in cam. 1 frame
- result:

$$\mathbf{R}^* = \mathbf{I}, \quad \mathbf{b}^* = \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{K}_1^* = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{K}_2^* = \begin{bmatrix} k_{21} & k_{22} & k_{23} \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (33)$$

- rectified cameras are in canonical relative pose $\quad$ not rotated, canonical baseline
- rectified calibration matrices can differ in the first row only
- when $\mathbf{K}_1^* = \mathbf{K}_2^*$ then the rectified pair is called the standard stereo pair and the homographies standard rectification homographies
- standard rectification homographies: points at infinity have zero disparity

$$\mathbf{P}_i^* \underline{\mathbf{X}}_\infty = \mathbf{K} \begin{bmatrix} \mathbf{I} & -\mathbf{C}_i \end{bmatrix} \underline{\mathbf{X}}_\infty = \mathbf{K} \mathbf{X}_\infty \qquad i = 1, 2$$

- this does not mean that the images are not distorted after rectification

▶Primitive Rectification

Goal: Given fundamental matrix $\mathbf{F}$, derive some simple rectification homographies $\mathbf{H}_1$, $\mathbf{H}_2$

1. Let the SVD of $\mathbf{F}$ be $\mathbf{U}\mathbf{D}\mathbf{V}^\top = \mathbf{F}$, where $\mathbf{D} = \operatorname{diag}(1, d^2, 0)$, $\quad 1 \geq d^2 > 0$
2. Write $\mathbf{D}$ as $\mathbf{D} = \mathbf{A}^\top \mathbf{F}^* \mathbf{B}$ for some regular $\mathbf{A}$, $\mathbf{B}$. For instance ($\mathbf{F}^*$ is given →155)

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -d & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & d & 0 \end{bmatrix}$$

3. Then

$$\mathbf{F} = \mathbf{U}\mathbf{D}\mathbf{V}^\top = \underbrace{\mathbf{U}\mathbf{A}^\top}_{\hat{\mathbf{H}}_2^\top} \mathbf{F}^* \underbrace{\mathbf{B}\mathbf{V}^\top}_{\hat{\mathbf{H}}_1} = \hat{\mathbf{H}}_2^\top \mathbf{F}^* \hat{\mathbf{H}}_1 \qquad \hat{\mathbf{H}}_1, \hat{\mathbf{H}}_2 \text{ orthonormal}$$

   and the primitive rectification homographies are

$$\hat{\mathbf{H}}_2 = \mathbf{A}\mathbf{U}^\top, \qquad \hat{\mathbf{H}}_1 = \mathbf{B}\mathbf{V}^\top$$

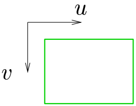
⊛ P1; 1pt: derive some other admissible $\mathbf{A}$, $\mathbf{B}$

- rectification homographies do exist →155
- there are other primitive rectification homographies, these suggested are just simple to obtain

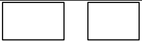
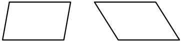
▶The Set of All Rectification Homographies

Proposition 1 Homographies $\mathbf{A}_1$ and $\mathbf{A}_2$ are rectification-preserving if the images stay rectified, i.e. if $\mathbf{A}_2^{-\top}\,\mathbf{F}^*\,\mathbf{A}_1^{-1} \simeq \mathbf{F}^*$, which gives

$$\mathbf{A}_1 = \begin{bmatrix} l_1 & l_2 & l_3 \\ 0 & s_v & t_v \\ 0 & q & 1 \end{bmatrix}, \qquad \mathbf{A}_2 = \begin{bmatrix} r_1 & r_2 & r_3 \\ 0 & s_v & t_v \\ 0 & q & 1 \end{bmatrix},$$



where $s_v \neq 0,\ t_v,\ l_1 \neq 0,\ l_2,\ l_3,\ r_1 \neq 0,\ r_2,\ r_3,\ q$ are 9 free parameters.

general	transformation		standard
$l_1,\ r_1$	horizontal scales		$l_1 = r_1$
$l_2,\ r_2$	horizontal shears		$l_2 = r_2$
$l_3,\ r_3$	horizontal shifts		$l_3 = r_3$
q	common special projective		
s_v	common vertical scale		
t_v	common vertical shift		
9 DoF			$9 - 3 = 6$ DoF

- q is due to a rotation about the baseline
- s_v changes the focal length

proof: find a rotation $\mathbf{G}$ that brings $\mathbf{K}$ to upper triangular form via RQ decomposition: $\mathbf{A}_1\mathbf{K}_1^* = \hat{\mathbf{K}}_1\mathbf{G}$ and $\mathbf{A}_2\mathbf{K}_2^* = \hat{\mathbf{K}}_2\mathbf{G}$

The Rectification Group

Corollary for Proposition 1 Let $\bar{\mathbf{H}}_1$ and $\bar{\mathbf{H}}_2$ be (primitive or other) rectification homographies. Then $\mathbf{H}_1 = \mathbf{A}_1\bar{\mathbf{H}}_1, \quad \mathbf{H}_2 = \mathbf{A}_2\bar{\mathbf{H}}_2$ are also rectification homographies.

Proposition 2 Pairs of rectification-preserving homographies $(\mathbf{A}_1, \mathbf{A}_2)$ form a group with group operation $(\mathbf{A}_1', \mathbf{A}_2') \circ (\mathbf{A}_1, \mathbf{A}_2) = (\mathbf{A}_1' \mathbf{A}_1, \mathbf{A}_2' \mathbf{A}_2)$.

Proof:

- closure by Proposition 1
- associativity by matrix multiplication
- identity belongs to the set
- inverse element belongs to the set by $\mathbf{A}_2^\top \mathbf{F}^* \mathbf{A}_1 \simeq \mathbf{F}^* \Leftrightarrow \mathbf{F}^* \simeq \mathbf{A}_2^{-\top} \mathbf{F}^* \mathbf{A}_1^{-1}$

▶Primitive Rectification Suffices for Calibrated Cameras

Obs: calibrated cameras: $d = 1 \Rightarrow \hat{\mathbf{H}}_1$, $\hat{\mathbf{H}}_2$ (→157) are orthonormal

1. determine primitive rectification homographies $(\hat{\mathbf{H}}_1, \hat{\mathbf{H}}_2)$ from the essential matrix
2. choose a suitable common calibration matrix $\mathbf{K}$, e.g.

$$\mathbf{K} = \begin{bmatrix} f & 0 & u_0 \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix}, \quad f = \frac{1}{2}(f^1 + f^2), \quad u_0 = \frac{1}{2}(u_0^1 + u_0^2), \quad \text{etc.}$$

3. the final rectification homographies applied as $\mathbf{P}_i \mapsto \mathbf{H}_i \mathbf{P}_i$ are

$$\mathbf{H}_1 = \mathbf{K}\hat{\mathbf{H}}_1\mathbf{K}_1^{-1}, \quad \mathbf{H}_2 = \mathbf{K}\hat{\mathbf{H}}_2\mathbf{K}_2^{-1}$$

- we got a standard stereo pair (→156) and non-negative disparity:

let $\mathbf{K}_i^{-1}\mathbf{P}_i = \mathbf{R}_i \begin{bmatrix} \mathbf{I} & -\mathbf{C}_i \end{bmatrix}, \quad i = 1, 2$ — note we started from $\mathbf{E}$, not $\mathbf{F}$

$$\mathbf{H}_1\mathbf{P}_1 = \mathbf{K}\hat{\mathbf{H}}_1\mathbf{K}_1^{-1}\mathbf{P}_1 = \mathbf{K}\underbrace{\mathbf{B}\mathbf{V}^\top\mathbf{R}_1}_{\mathbf{R}^*}\begin{bmatrix} \mathbf{I} & -\mathbf{C}_1 \end{bmatrix} = \mathbf{K}\mathbf{R}^*\begin{bmatrix} \mathbf{I} & -\mathbf{C}_1 \end{bmatrix}$$

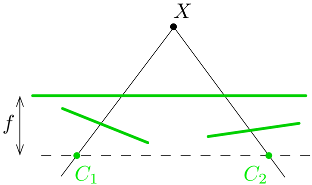
$$\mathbf{H}_2\mathbf{P}_2 = \mathbf{K}\hat{\mathbf{H}}_2\mathbf{K}_2^{-1}\mathbf{P}_2 = \mathbf{K}\underbrace{\mathbf{A}\mathbf{U}^\top\mathbf{R}_2}_{\mathbf{R}^*}\begin{bmatrix} \mathbf{I} & -\mathbf{C}_2 \end{bmatrix} = \mathbf{K}\mathbf{R}^*\begin{bmatrix} \mathbf{I} & -\mathbf{C}_2 \end{bmatrix}$$

- one can prove that $\mathbf{B}\mathbf{V}^\top\mathbf{R}_1 = \mathbf{A}\mathbf{U}^\top\mathbf{R}_2$ with the help of essential matrix decomposition (13)
- points at infinity project by $\mathbf{K}\mathbf{R}^*$ in both cameras ⇒ they have zero disparity

→166

►Summary & Remarks: Linear Rectification

standard rectification homographies reproject onto a common image plane parallel to the baseline



- rectification is done with a pair of homographies (one per image) →154
  - ⇒ rectified camera centers are equal to the original ones
  - binocular rectification: a 9-parameter family of rectification homographies
  - trinocular rectification: has 9 or 6 free parameters (depending on additional constrains)
  - in general, linear rectification is not possible for more than three cameras
- rectified cameras are in canonical orientation →156
  - ⇒ rectified image projection planes are coplanar
- equal rectified calibration matrices give standard rectification →156
  - ⇒ rectified image projection planes are equal
- primitive rectification is already standard in calibrated cameras →160

- known $\mathbf{F}$ used alone does not allow standardization of rectification homographies
- for that we need either of these:
  1. projection matrices, or calibrated cameras, or
  2. a few points at infinity calibrating k_{1i}, k_{2i}, $i = 1, 2, 3$ in (33)

Optimal and Non-linear Rectification

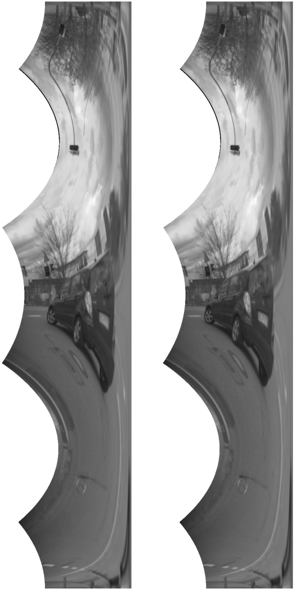
Optimal choice for the free parameters

- by minimization of residual image distortion, eg. [Gluckman & Nayar 2001]

$$\mathbf{A}_1^* = \arg\min_{\mathbf{A}_1} \iint_\Omega \left(\det J(\mathbf{A}_1 \hat{\mathbf{H}}_1 \underline{\mathbf{x}}) - 1\right)^2 d\mathbf{x}$$

- by minimization of image information loss [Matoušek, ICIG 2004]
- non-linear rectification suitable for forward motion
  non-parametric: [Pollefeys et al. 1999]
  analytic: [Geyer & Daniilidis 2003]

forward egomotion



rectified images, Pollefeys' method

Thank You