# 3D Computer Vision

Radim Šára     Martin Matoušek

Center for Machine Perception
Department of Cybernetics
Faculty of Electrical Engineering
Czech Technical University in Prague

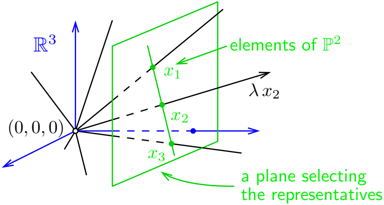
https://cw.fel.cvut.cz/wiki/courses/tdv/start

http://cmp.felk.cvut.cz
mailto:sara@cmp.felk.cvut.cz
phone ext. 7203

rev. October 5, 2021

Open Informatics Master's Course

# ▶Homography in $\mathbb{P}^2$




Projective plane $\mathbb{P}^2$: Vector space of dimension 3 excluding the zero vector, $\mathbb{R}^3 \setminus (0,0,0)$, factorized to linear equivalence classes ('rays'), $\underline{\mathbf{x}} \simeq \lambda \underline{\mathbf{x}}$, $\lambda \neq 0$

including 'points at infinity'

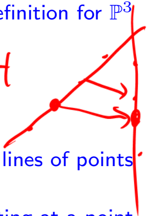
we call $\underline{\mathbf{x}} \in \mathbb{P}^2$ 'points'

**Homography in $\mathbb{P}^2$:** Non-singular linear mapping in $\mathbb{P}^2$

an analogic definition for $\mathbb{P}^3$

$$\underline{\mathbf{x}}' \simeq \mathbf{H}\, \underline{\mathbf{x}}, \quad \mathbf{H} \in \mathbb{R}^{3,3} \text{ non-singular}$$

$\lambda H \simeq H$



## Defining properties

- collinear points are mapped to collinear points

  lines of points are mapped to lines of points

- concurrent lines are mapped to concurrent lines

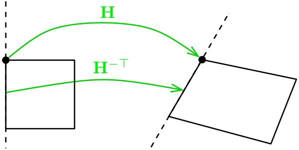
  concurrent = intersecting at a point

- and point-line incidence is preserved

  e.g. line intersection points mapped to line intersection points

- $\mathbf{H}$ is a $3 \times 3$ non-singular matrix, $\lambda\, \mathbf{H} \simeq \mathbf{H}$ equivalence class, 8 degrees of freedom
- homogeneous matrix representative: $\det \mathbf{H} = 1$ $\qquad \mathbf{H} \in \mathrm{SL}(3)$
- what we call homography here is often called 'projective collineation' in mathematics

# ▶Mapping 2D Points and Lines by Homography




$$\underline{\mathbf{m}}' \simeq \mathbf{H}\,\underline{\mathbf{m}} \qquad \text{(image) point}$$

$$\underline{\mathbf{n}}' \simeq \mathbf{H}^{-\top}\underline{\mathbf{n}} \qquad \text{(image) line} \qquad \mathbf{H}^{-\top} = (\mathbf{H}^{-1})^{\top} = (\mathbf{H}^{\top})^{-1}$$

- incidence is preserved: $(\underline{\mathbf{m}}')^{\top}\underline{\mathbf{n}}' \simeq \underline{\mathbf{m}}^{\top}\mathbf{H}^{\top}\mathbf{H}^{-\top}\underline{\mathbf{n}} = \underline{\mathbf{m}}^{\top}\underline{\mathbf{n}} = 0$

**Mapping a finite 2D point $\mathbf{m} = (u, v)$ to $\underline{\mathbf{m}} = (u', v')$**

1. extend the Cartesian (pixel) coordinates to homogeneous coordinates, $\underline{\mathbf{m}} = (u, v, \mathbf{1})$
2. map by homography, $\underline{\mathbf{m}}' = \mathbf{H}\,\underline{\mathbf{m}}$
3. if $m_3' \neq 0$ convert the result $\underline{\mathbf{m}}' = (m_1', m_2', m_3')$ back to Cartesian coordinates (pixels),

$$u' = \frac{m_1'}{m_3'}\,\mathbf{1}, \qquad v' = \frac{m_2'}{m_3'}\,\mathbf{1}$$

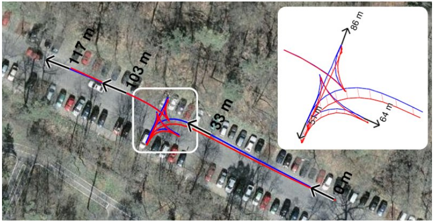
- note that, typically, $m_3' \neq 1$ — $m_3' = 1$ when $\mathbf{H}$ is affine
- an infinite point $\underline{\mathbf{m}} = (u, v, 0)$ maps the same way

# Some Homographic Tasters

**Rectification of camera rotation**: →53 (geometry), →122 (homography estimation)


$\mathbf{H} \simeq \mathbf{K}\mathbf{R}^\top\mathbf{K}^{-1}$ maps from image plane to facade plane

**Homographic Mouse for Visual Odometry**: [Mallis 2007]


illustrations courtesy of AMSL Racing Team, Meiji University and LIBVISO: Library for VISual Odometry

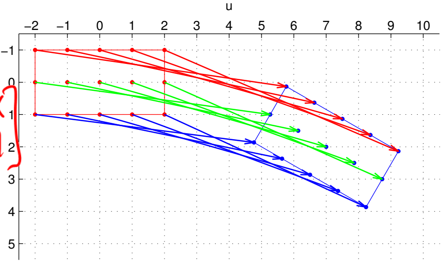
$\mathbf{H} \simeq \mathbf{K}\left(\mathbf{R} - \frac{\mathbf{t}\mathbf{n}^\top}{d}\right)\mathbf{K}^{-1}$ maps from plane to translated plane [H&Z, p. 327]

# ▶Homography Subgroups: Euclidean Mapping (aka Rigid Motion)

- Euclidean mapping (EM): rotation, translation and their combination

$$\mathbf{H} = \begin{bmatrix} \cos\phi & -\sin\phi & t_x \\ \sin\phi & \cos\phi & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

- eigenvalues $(1, e^{-i\phi}, e^{i\phi})$ $Rx + t$
- $\mathbf{H} \in \mathrm{SE}(3)$




rotation by $30^\circ$, then translation by $(7,\,2)$

**EM = The most general homography preserving**

1. **lengths**: Let $\underline{\mathbf{x}}'_i = \mathbf{H}\underline{\mathbf{x}}_i$ (check we can use $=$ instead of $\simeq$). Let $(x_i)_3 = 1$, Then

$$\|\underline{\mathbf{x}}'_2 - \underline{\mathbf{x}}'_1\| = \|\mathbf{H}\underline{\mathbf{x}}_2 - \mathbf{H}\underline{\mathbf{x}}_1\| = \|\mathbf{H}(\underline{\mathbf{x}}_2 - \underline{\mathbf{x}}_1)\| \overset{\circledast\ \text{P1; 1pt}}{=} \cdots = \|\underline{\mathbf{x}}_2 - \underline{\mathbf{x}}_1\|$$

2. **angles** check the dot-product of normalized differences from a point $(\mathbf{x} - \mathbf{z})^\top(\mathbf{y} - \mathbf{z})$ (Cartesian(!))
3. **areas**: $\det \mathbf{H} = 1 \;\Rightarrow\;$ unit Jacobian; follows from 1. and 2.

- eigenvectors when $\phi \neq k\pi,\ k = 0, 1, \ldots$ (columnwise)

$$\mathbf{e}_1 \simeq \begin{bmatrix} t_x + t_y \cot\frac{\phi}{2} \\ t_y - t_x \cot\frac{\phi}{2} \\ 2 \end{bmatrix}, \quad \mathbf{e}_2 \simeq \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 \simeq \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}$$

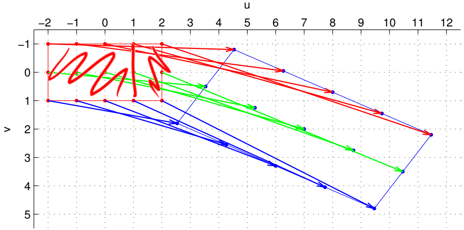
$\mathbf{e}_2, \mathbf{e}_3$ – circular points, $i$ – imaginary unit

4. **circular points**: points at infinity $(i, 1, 0)$, $(-i, 1, 0)$ (preserved even by similarity)

- **similarity**: scaled Euclidean mapping (does not preserve lengths, areas)

# ▶Homography Subgroups: Affine Mapping

$$\mathbf{H} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$




rotation by $30^\circ$
then scaling by $\mathrm{diag}(1,\ 1.5,\ 1)$
then translation by $(7,\ 2)$

**AM = The most general homography preserving**

- parallelism
- ratio of areas
- ratio of lengths on parallel lines
- linear combinations of vectors (e.g. midpoints)
- convex hull
- line at infinity $\underline{\mathbf{n}}_\infty$ (not pointwise)

**does not preserve**

- lengths
- angles
- areas
- circular points

observe $\mathbf{H}^\top \underline{\mathbf{n}}_\infty \simeq \begin{bmatrix} a_{11} & a_{21} & 0 \\ a_{12} & a_{22} & 0 \\ t_x & t_y & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underline{\mathbf{n}}_\infty \quad \Rightarrow \quad \underline{\mathbf{n}}_\infty \simeq \mathbf{H}^{-\top} \underline{\mathbf{n}}_\infty$

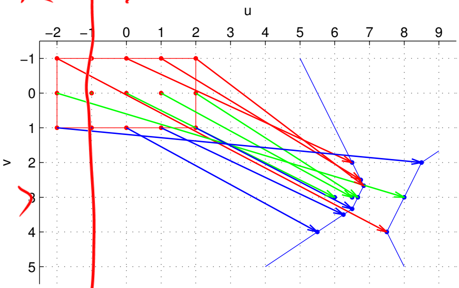
Euclidean mappings preserve all properties affine mappings preserve, of course

# ▶Homography Subgroups: General Homography

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \qquad \mathbf{H} \in \mathrm{SL}(3)$$

u

v




$$\mathbf{H} = \begin{bmatrix} 7 & -0.5 & 6 \\ 3 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

line $\underline{\mathbf{n}} = (1, 0, 1)$ is mapped to $\underline{\mathbf{n}}_\infty$: $\mathbf{H}^{-\top}\underline{\mathbf{n}} \simeq \underline{\mathbf{n}}_\infty$

(where in the picture is the line $n$?)

## preserves only

- incidence and concurrency
- collinearity
- cross-ratio on the line →46

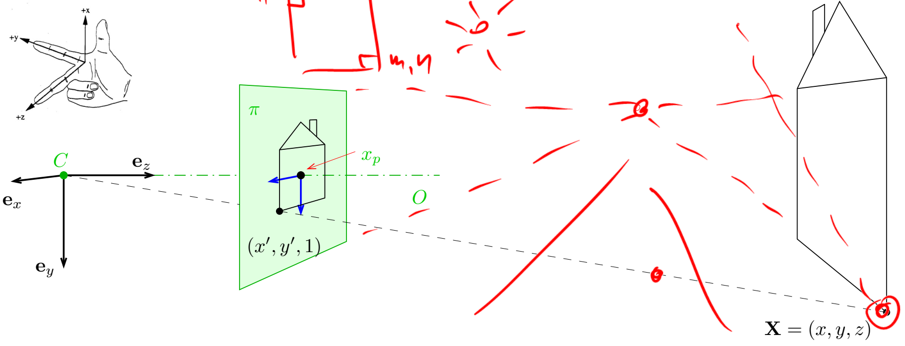
## does not preserve

- lengths
- areas
- parallelism
- ratio of areas
- ratio of lengths
- linear combinations of vectors (midpoints, etc.)
- convex hull
- line at infinity $\underline{\mathbf{n}}_\infty$

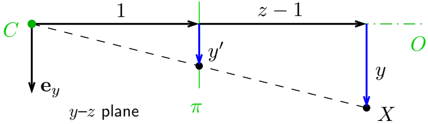
H = vandn(3,3)

det H ≠ 0

# ▸Canonical Perspective Camera (Pinhole Camera, Camera Obscura)




1. in this picture we are looking 'down the street'
2. right-handed canonical coordinate system $(x, y, z)$ with unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$
3. origin = center of projection $C$
4. image plane $\pi$ at unit distance from $C$
5. optical axis $O$ is perpendicular to $\pi$
6. principal point $x_p$: intersection of $O$ and $\pi$
7. perspective camera is given by $C$ and $\pi$




projected point in the natural image coordinate system:

$$\frac{y'}{1} = y' = \frac{y}{1+z-1} = \frac{y}{z}, \qquad x' = \frac{x}{z}$$

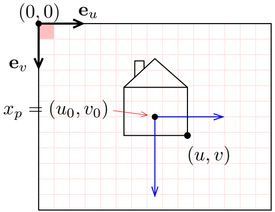
projected point **in canonical camera** $(z \neq 0)$

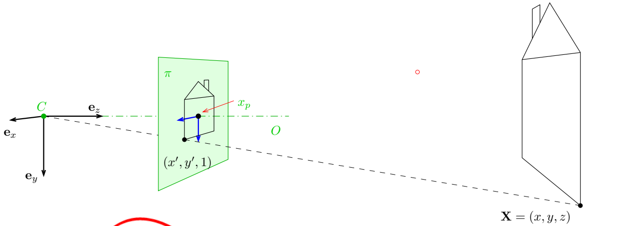
$$(x', y', 1) = \left(\frac{x}{z}, \frac{y}{z}, 1\right) = \frac{1}{z}(x, y, z) \simeq (x, y, z) = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{P}_0 = \left[\mathbf{I} \mid \mathbf{0}\right]} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{P}_0 \, \underline{\mathbf{X}}$$

$\lambda \neq 0$

projected point **in scanned image**

scale by $f$ and translate origin to image corner




$$u = f\,\frac{x}{z} + u_0 \qquad v = f\,\frac{y}{z} + v_0$$

$$\frac{1}{z}\begin{bmatrix} f\,x + z\,u_0 \\ f\,y + z\,v_0 \\ z \end{bmatrix} \simeq \begin{bmatrix} f & 0 & u_0 \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{K}\mathbf{P}_0 \, \underline{\mathbf{X}} = \mathbf{P} \, \underline{\mathbf{X}}$$

- 'calibration' matrix $\mathbf{K}$ transforms canonical $\mathbf{P}_0$ to standard perspective camera $\mathbf{P}$

# ▶Computing with Perspective Camera Projection Matrix

Projection from world to image in standard camera $\mathbf{P}$:

$$\underbrace{\begin{bmatrix} f & 0 & u_0 & 0 \\ 0 & f & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{P} \; = K P} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} f x + u_0 z \\ f y + v_0 z \\ z \end{bmatrix} \simeq \underbrace{\begin{bmatrix} x + \frac{z}{f} u_0 \\ y + \frac{z}{f} v_0 \\ \frac{z}{f} \end{bmatrix}}_{\text{(a)}} \simeq \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \underline{\mathbf{m}}$$

$$\frac{m_1}{m_3} = \frac{f\,x}{z} + u_0 = u, \qquad \frac{m_2}{m_3} = \frac{f\,y}{z} + v_0 = v \quad \text{when} \quad m_3 \neq 0$$

$f$ – 'focal length' – converts length ratios to pixels, $[f] = \mathrm{px}$, $f > 0$

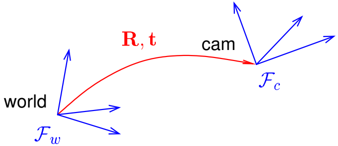
$(u_0, v_0)$ – principal point in pixels

**Perspective Camera:**

1. dimension reduction — since $\mathbf{P} \in \mathbb{R}^{3,4}$
2. nonlinear unit change $\mathbf{1} \mapsto \mathbf{1} \cdot z/f$, see (a)
   for convenience we use $P_{11} = P_{22} = f$ rather than $P_{33} = 1/f$ and the $u_0$, $v_0$ in relative units
3. $m_3 = 0$ represents points at infinity in image plane $\pi$ — i.e. points with $z = 0$

# ▸Changing The Outer (World) Reference Frame

A transformation of a point from the world to camera coordinate system:




$$\mathbf{X}_c = \mathbf{R}\,\mathbf{X}_w + \mathbf{t}$$

$\mathbf{R}$ – camera rotation matrix — world orientation in the camera coordinate frame $\mathcal{F}_c$

$\mathbf{t}$ – camera translation vector — world origin in the camera coordinate frame $\mathcal{F}_c$

$$\mathbf{P}\,\underline{\mathbf{X}}_c = \mathbf{K}\mathbf{P}_0 \begin{bmatrix} \mathbf{X}_c \\ 1 \end{bmatrix} = \mathbf{K}\mathbf{P}_0 \begin{bmatrix} \mathbf{R}\mathbf{X}_w + \mathbf{t} \\ 1 \end{bmatrix} = \mathbf{K} \underbrace{\begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}}_{P_0} \underbrace{\begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}}_{\mathbf{T}} \begin{bmatrix} \mathbf{X}_w \\ 1 \end{bmatrix} = \mathbf{K} \underbrace{\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}}_{P} \underline{\mathbf{X}}_w$$

$\mathbf{P}_0$ (a $3 \times 4$ mtx) discards the last row of $\mathbf{T}$

- $\mathbf{R}$ is rotation, $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$, $\det \mathbf{R} = +1$ — $\mathbf{I} \in \mathbb{R}^{3,3}$ identity matrix
- 6 **extrinsic parameters**: 3 rotation angles (Euler theorem), 3 translation components
- alternative, often used, camera representations

$$\mathbf{P} = \mathbf{K}\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} = \mathbf{K}\mathbf{R}\begin{bmatrix} \mathbf{I} & -\mathbf{C} \end{bmatrix}$$

$P \simeq \lambda P \quad \lambda \neq 0$

$\mathbf{C}$ – camera position in the world reference frame $\mathcal{F}_w$ — $\mathbf{t} = -\mathbf{R}\mathbf{C}$

$\mathbf{r}_3^\top$ – optical axis in the world reference frame $\mathcal{F}_w$ — third row of $\mathbf{R}$: $\mathbf{r}_3 = \mathbf{R}^{-1}[0, 0, 1]^\top$

- we can save some conversion and computation by noting that $\mathbf{K}\mathbf{R}\begin{bmatrix} \mathbf{I} & -\mathbf{C} \end{bmatrix} \underline{\mathbf{X}} = \mathbf{K}\mathbf{R}(\mathbf{X} - \mathbf{C})$

# ►Changing the Inner (Image) Reference Frame

**The general form of calibration matrix $\mathbf{K}$ includes**

- skew angle $\theta$ of the digitization raster
- pixel aspect ratio $a$

$$\mathbf{K} = \begin{bmatrix} a\,f & -a\,f\,\cot\theta & u_0 \\ 0 & f/\sin\theta & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

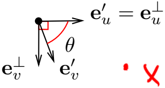
units: $[f] = \mathrm{px}$, $[u_0] = \mathrm{px}$, $[v_0] = \mathrm{px}$, $[a] = 1$

⊛ H1; 2pt: Decompose $\mathbf{K}$ to simple maps and give $f, a, \theta, u_0, v_0$ a precise meaning; deadline LD+2 wk

Hints:

1. image projects to orthogonal system $F^{\perp}$, then it maps by skew to $F'$, then by scale $a\,f$, $f$ to $F''$, then by translation by $u_0$, $v_0$ to $F'''$
2. Skew: Express point $\mathbf{x}$ as

$$\mathbf{x} = u'\mathbf{e}_{u'} + v'\mathbf{e}_{v'} = u^{\perp}\mathbf{e}_u^{\perp} + v^{\perp}\mathbf{e}_v^{\perp}$$

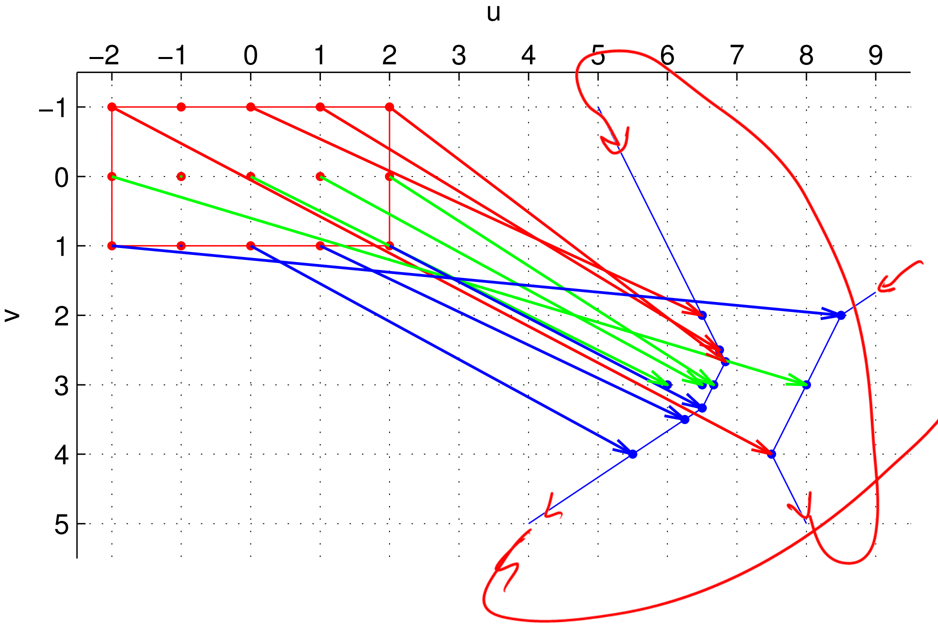



   $\mathbf{e}_:$ are unit basis vectors

3. $\mathbf{K}$ maps from $F^{\perp}$ to $F'''$ as

$$w'''\,[u''', v''', 1]^{\top} = \mathbf{K}[u^{\perp}, v^{\perp}, 1]^{\top}$$

Thank You


u
−2
−1
0
1
2
3
4
5
6
7
8
9
v
−1
0
1
2
3
4
5