# Test 2 - Solution

December 11, 2021

**Task 1.** *Consider motion given by a mapping of a general point $X$ to point $Y$ by*

$$\vec{y}_\beta = \mathtt{R}\,\vec{x}_\beta + \vec{o}\,'_\beta\,,$$

*where $\vec{x}_\beta$, resp. $\vec{y}_\beta$, are coordinate vectors representing point $X$, resp. point $Y$, in a coordinate system with an orthonormal basis $\beta$. Matrix $\mathtt{R}$ and vector $\vec{o}\,' = \overrightarrow{OO'}$ are given as follows*

$$\mathtt{R} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad \vec{o}\,'_\beta = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

1. *Write down a matrix equation determining the coordinates of points on the axis of motion.*
2. *Find all the points on the axis of motion.*

**Solution:** The matrix equation which determines the points on the axis of motion looks as follows:

$$(\mathtt{R} - \mathtt{I})^2 \vec{x}_\beta = -(\mathtt{R} - \mathtt{I})\vec{o}\,'_\beta$$

Substituting $\mathtt{R}$ and $\vec{o}\,'_\beta$ to it we obtain

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}^2 \vec{x}_\beta = -\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{bmatrix} \vec{x}_\beta = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

We can solve this system of linear equations by Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \\ -2 & 1 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From the last matrix we can see that the particular solution is $(1, 0, 0)$ and the solutions of the corresponding homogeneous system are $\{\alpha \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top \mid \alpha \in \mathbb{R}\}$. That's why the solutions to the inhomogeneous system are:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \;\middle|\; \alpha \in \mathbb{R} \right\}.$$

□

**Task 2.** *Consider the following unit quaternion*

$$\mathbf{q} = \left[\frac{4}{5}, \frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right]^\top$$

1. *For the rotation given by $\mathbf{q}$, find all pairs $[\theta, \mathbf{v}]$ of the corresponding rotation angle $-\pi < \theta \leq \pi$ and rotation axis represented by a unit vector $\mathbf{v}$.*
2. *Construct the corresponding rotation matrix.*

**Solution:** We know that

$$\mathbf{q} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\mathbf{v} \end{bmatrix}$$

where $\theta$ is the angle of rotation and $\mathbf{v}$ is the normalized axis of rotation. That's why

$$\cos\frac{\theta}{2} = \frac{4}{5} \Rightarrow \sin\frac{\theta}{2} = \pm\frac{3}{5}$$

We first take the pair $(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}) = (\frac{4}{5}, \frac{3}{5})$ which gives

$$\theta = 2 \cdot \operatorname{atan2}\left(\frac{3}{5}, \frac{4}{5}\right).$$

We compute the normalized axis of rotation $\mathbf{v}$ by dividing the last 3 coordinates of $\mathbf{q}$ by $\sin\frac{\theta}{2}$:

$$\mathbf{v} = \frac{1}{\frac{3}{5}} \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{bmatrix}^\top = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}^\top.$$

After we've found the first pair $(\theta, \mathbf{v})$ which gives $\mathbf{q}$, all the pairs $(\theta, \mathbf{v})$ which give the rotation defined by $\mathbf{q}$ are given by $\{(\theta, \mathbf{v}), (-\theta, -\mathbf{v})\}$. Hence, the answer is

$$\left\{ \left(2 \cdot \operatorname{atan2}\left(\frac{3}{5}, \frac{4}{5}\right), \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}^\top\right), \left(-2 \cdot \operatorname{atan2}\left(\frac{3}{5}, \frac{4}{5}\right), -\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}^\top\right) \right\}.$$

□

**Task 3.** *Consider the rotation with rotation axis generated by vector* $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$ *that maps vector* $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$ *to vector* $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top$.

1. *Find its rotation angle* $-\pi < \theta \leq \pi$.
2. *Find its rotation matrix* $\mathtt{R}$.
3. *Find the eigenvalues of* $\mathtt{R}$.

**Solution:** We use the angle-axis parametrization of the rotation:

$$\mathtt{R} = \cos\theta\mathtt{I} + (1 - \cos\theta)\mathbf{v}\mathbf{v}^\top + \sin\theta[\mathbf{v}]_\times \tag{1}$$

where $\mathbf{v} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ is the normalized axis of rotation. By the task, $\mathtt{R}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top$. Hence, multiplying both sides of Equation (1) by $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$ we get

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathtt{R}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \left(\cos\theta\mathtt{I} + (1 - \cos\theta)\mathbf{v}\mathbf{v}^\top + \sin\theta[\mathbf{v}]_\times\right)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \cos\theta\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{1 - \cos\theta}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{\sin\theta}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \tag{2}$$

From the last equation of Equation (2)

$$0 = \cos\theta + \frac{1 - \cos\theta}{3}$$

we can express

$$\cos\theta = -\frac{1}{2}.$$

Substituting it to the second equation in (2) we get

$$0 = \frac{1}{2} - \frac{\sin\theta}{\sqrt{3}}$$

from which we get

$$\sin\theta = \frac{\sqrt{3}}{2}$$

The rotation angle then equals

$$\theta = \operatorname{atan2}(\sin\theta, \cos\theta) = \operatorname{atan2}\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \frac{2\pi}{3}.$$

We get the rotation matrix by substituting $\mathbf{v}$ and $\theta$ to Equation (1):

$$\mathtt{R} = -\frac{1}{2}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0 & -1 & 1\\ 1 & 0 & -1\\ -1 & 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$$

The eigenvalues of $\mathtt{R}$ are the roots of the characteristic polynomial of $\mathtt{R}$:

$$p(\lambda) = \det(\lambda\mathtt{I} - \mathtt{R}) = \lambda^3 - 1$$

whose roots are the cubic roots of unity $1, e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{2}{3}}$. □

**Task 4.** *Consider the following rotation matrix*

$$\mathtt{R} = \frac{1}{3}\begin{bmatrix}-1 & 2 & 2\\ 2 & -1 & 2\\ 2 & 2 & -1\end{bmatrix}$$

*Find all unit quaternions that represent* $\mathtt{R}$.

**Solution:** The given rotation matrix is symmetric, meaning that it is a rotation by $\pi$. That's why the rotation angle $\theta = \pi$, from which we get

$$\cos\frac{\theta}{2} = \cos\frac{\pi}{2} = 0, \quad \sin\frac{\theta}{2} = \sin\frac{\pi}{2} = 1.$$

The normalized rotation axis can be computed by finding the kernel of $\mathtt{R} - \mathtt{I}$:

$$\mathtt{R} - \mathtt{I} = \frac{1}{3}\begin{bmatrix}-4 & 2 & 2\\ 2 & -4 & 2\\ 2 & 2 & -4\end{bmatrix}$$

Applying Gaussian elimination on $\mathtt{R} - \mathtt{I}$ we get:

$$\frac{1}{3}\begin{bmatrix}-4 & 2 & 2\\ 2 & -4 & 2\\ 2 & 2 & -4\end{bmatrix} \sim \begin{bmatrix}-4 & 2 & 2\\ 2 & -4 & 2\\ 2 & 2 & -4\end{bmatrix} \sim \begin{bmatrix}-4 & 2 & 2\\ 0 & -3 & 3\\ 0 & 3 & -3\end{bmatrix} \sim \begin{bmatrix}-4 & 2 & 2\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$$

from which we get the kernel

$$\left\{\alpha\begin{bmatrix}1\\1\\1\end{bmatrix} \,\middle|\, \alpha \in \mathbb{R}\right\}.$$

The normalized rotation axis is then

$$\mathbf{v} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

Using the formula for the quaternion

$$\mathbf{q} = \begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\mathbf{v}\end{bmatrix}$$

we substitute $\theta$ and $\mathbf{v}$ to it and get

$$\mathbf{q}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}0\\1\\1\\1\end{bmatrix}.$$

Since there are always two pairs of unit quaterions with opposite sings which generate the same rotation (except for the identity rotation), then all the unit quaternions which define `R` are

$$\left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, -\frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

□