Mathematical Formulation of IKT

$$M = M_1^0 M_2^1 M_3^2 M_4^3 M_5^4 M_6^5$$

$$\underbrace{\begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}}_{\substack{\text{pose of the} \\ \text{end effector}}} = \prod_{i=1}^{6} M_i^{i-1}(\theta_i + \underbrace{\theta_{i_{\text{offset}}}, d_i, a_i, \alpha_i}_{\text{DH parameters}})$$

$$\begin{bmatrix} f_1 & \dots & f_4 \\ \vdots & \ddots & \vdots \\ f_9 & \dots & f_{12} \\ 0 & \dots & 0 \end{bmatrix} = \prod_{i=1}^{6} M_i^{i-1}(\theta_i + \theta_{i_{\text{offset}}}, d_i, a_i, \alpha_i) - \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} = \mathbf{O}$$

Symbolic formulation

Every matrix M_i^{i-1} can be decomposed as

$$\begin{bmatrix} \cos(\theta_i + \theta_{i_{\text{offset}}}) & -\sin(\theta_i + \theta_{i_{\text{offset}}}) & 0 & 0 \\ \sin(\theta_i + \theta_{i_{\text{offset}}}) & \cos(\theta_i + \theta_{i_{\text{offset}}}) & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

New variables:

$$c_i = \cos(\theta_i + \theta_{i_{\text{offset}}}), \quad s_i = \sin(\theta_i + \theta_{i_{\text{offset}}})$$

New polynomial equations:

$$\prod_{i=1}^{6} M_i^{i-1}(c_i, s_i, d_i, a_i, \alpha_i) - \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} = \mathbf{O}$$

$$c_i{}^2 + s_i{}^2 = 1, \quad i = 1, \ldots, 6$$

Compute θ_i from c_i and s_i:

$$\theta_i = \operatorname{atan2}(s_i, c_i) - \theta_{i_{\text{offset}}}$$

Simplified equations

Let's look at the inverse matrix $\left(M_i^{i-1}\right)^{-1}$

$$\begin{bmatrix} 1 & 0 & 0 & -a_i \\ 0 & \cos\alpha_i & \sin\alpha_i & 0 \\ 0 & -\sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_i & s_i & 0 & 0 \\ -s_i & c_i & 0 & 0 \\ 0 & 0 & 1 & -d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since it is polynomial in c_i and s_i, thus the old polynomial equations of IKT of degree 6 in c_i and s_i can be simplified to the following ones of degree 3:

$$\prod_{i=4}^{6} M_i^{i-1}(c_i, s_i, d_i, a_i, \alpha_i) - \left(\prod_{i=1}^{3} M_i^{i-1}(c_i, s_i, d_i, a_i, \alpha_i)\right)^{-1} \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} = \mathbf{O},$$

$$c_i{}^2 + s_i{}^2 = 1, \quad i = 1, \ldots, 6$$

Solvability of equations

1. Groebner Basis computation is done in exact arithmetics over rational numbers and therefore input must be provided in rational numbers.
2. At the same time, the input must satisfy all identities on sines, cosines and rotations, otherwise the systems would have no solution.

That's why,

1. the parameters d_i, a_i must be given as `Fraction` numbers, the translation $\mathbf{t}$ must be given as a vector with `Fraction` numbers
2. $\cos\alpha_i$ and $\sin\alpha_i$ must be given as a tuple of `Fraction` numbers such that the sum of their squares is exactly 1
3. the rotation matrix $\mathbf{R}$ must be given as a matrix with `Fraction` numbers such that $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$ and $\det\mathbf{R} = 1$ hold exactly.

Rational approximation of a floating point number

Input: floating point number n, positive tolerance tol
Output: fraction number f such that $|f - n| < tol$
Steps:

1. If $tol \geq 1$: create a rational approximation f of n as:

$$f = \frac{\lfloor n \rfloor}{1}$$

2. If $tol < 1$: represent tol in scientific notation and take the exponent e, i.e.

$$tol = m \times 10^e, \quad m \in [1, 10), \ e \in \mathbb{Z}_{<0}$$

and create a rational approximation f of n as:

$$f = \frac{\lfloor n \cdot 10^{-e} \rfloor}{10^{-e}}$$

Rational approximation of a floating point number

Example

Let the floating point number be given by

$$n = 10.123456789$$

and the tolerance of the rational approximation be given by another floating point number

$$tol = 0.00002 5932 = 2.5932 \times 10^{-5} \Rightarrow e = -5$$

The fraction which approximates n is

$$f = \frac{\lfloor n \cdot 10^{-e} \rfloor}{10^{-e}} = \frac{\lfloor 10.123456789 \cdot 10^{5} \rfloor}{10^{5}} = \frac{1012345}{10^{5}}$$

$$|f - n| = |10.12345 - 10.123456789| = 0.000006789 < tol$$

Exact sine and cosine

Rational parametrization of the unit circle:

$$\cos\theta = \frac{1-t^2}{1+t^2}, \quad \sin\theta = \frac{2t}{1+t^2}, \quad t \in \mathbb{Q}$$

Trigonometric meaning of t:

$$\tan\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{\sin\theta} = \frac{1-\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}} = \frac{2t^2}{2t} = t$$

Exact sine and cosine

Input: floating point number $\theta \in [-\pi, \pi]$, positive tolerance tol
Output: fraction numbers c, s such that

$$c^2 + s^2 = 1 \text{ (exactly)} \quad \& \quad \exists k \in \mathbb{Z} : |\text{atan2}(s, c) - \theta + 2k\pi| < tol$$

Steps:

1. If $|\theta - \pi| < tol$ or $|\theta + \pi| < tol$, then return

$$c = -1, \quad s = 0$$

2. Compute tangent of $\frac{\theta}{2}$ and approximate it by rational fraction:

$$t = \texttt{rat\_approx}\left(\tan\left(\frac{\theta}{2}\right), \frac{tol}{10}\right)$$

3. Compute cosine and sine corresponding to t:

$$c = \frac{1 - t^2}{1 + t^2}, \quad s = \frac{2t}{1 + t^2}$$

Exact sine and cosine

Example

Let the angle and the tolerance be given by

$$\theta = 1.2345, \quad tol = 0.0023$$

The rational approximation of $\tan\left(\frac{\theta}{2}\right)$ is given by

$$t = \texttt{rat\_approx}\left(\tan\left(\frac{\theta}{2}\right), tol\right) = \texttt{rat\_approx}\left(0.709766, 0.0023\right) =$$

$$= \frac{\lfloor 0.709766 \cdot 10^3 \rfloor}{10^3} = \frac{709}{1000}$$

$$c = \frac{1-t^2}{1+t^2} = \frac{497319}{1502681}, \quad s = \frac{2t}{1+t^2} = \frac{1418000}{1502681}$$

$$|\mathrm{atan2}(s,c) - \theta| = |1.23348 - 1.2345| = 0.00102 < tol$$

Exact 3×3 rotation matrix

Parametrization of exact 3×3 rotation matrices:

$$\mathbf{R} = \frac{1}{\sum_{i=1}^{4} q_i^2} \begin{bmatrix} q_1^2 + q_2^2 - q_3^2 - q_4^2 & 2(q_2 q_3 - q_1 q_4) & 2(q_2 q_4 + q_1 q_3) \\ 2(q_2 q_3 + q_1 q_4) & q_1^2 - q_2^2 + q_3^2 - q_4^2 & 2(q_3 q_4 - q_1 q_2) \\ 2(q_2 q_4 - q_1 q_3) & 2(q_3 q_4 + q_1 q_2) & q_1^2 - q_2^2 - q_3^2 + q_4^2 \end{bmatrix},$$

$$q_1, q_2, q_3, q_4 \in \mathbb{Q}$$

In the next slide we will use the function `q2r` which is defined by the above formula.

Exact 3×3 rotation matrix

Algorithm 1: Exact 3×3 rotation matrix

Input: vector of float numbers $\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}^\top$ with $\|\mathbf{q}\| \approx 1$, positive tolerance tol

Output: 3×3 matrix $\mathbf{R}$ with fraction numbers such that

$$\mathbf{R}^\top \mathbf{R} = \mathrm{I}, \;\; \det \mathbf{R} = 1 \text{ (exactly)} \quad \& \quad ||\mathbf{R} - \mathtt{q2r}(q)||_\mathrm{F} < tol$$

$tol_q \leftarrow tol$

while $TRUE$ **do**

- $q_{rat} \leftarrow \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$
- **for** $i \leftarrow 1$ **to** 4 **do**
  - $q_{rat}[i] \leftarrow \mathtt{rat\_approx}(q_i, tol_q)$
- $\mathbf{R} \leftarrow \mathtt{q2r}(q_{rat})$
- **if** $\|\mathbf{R} - \mathtt{q2r}(q)\|_\mathrm{F} < tol$ **then**
  - **return** $\mathbf{R}$
- **else**
  - $tol_q \leftarrow \frac{tol_q}{10}$

Exact 3×3 rotation matrix

Example

Let the quaternion and the tolerance be given by

$$\mathbf{q} = \begin{bmatrix} 0.748 & 0.654 & 0.108 & 0.012 \end{bmatrix}, \quad tol = 0.0011$$

The output of the algorithm `exact_rot` gives the exact rotation matrix

$$\mathbf{R} = \begin{bmatrix} \frac{243853}{249757} & \frac{30828}{249757} & \frac{44316}{249757} \\ \frac{39804}{249757} & \frac{35827}{249757} & -\frac{243948}{249757} \\ -\frac{36468}{249757} & \frac{245244}{249757} & \frac{30067}{249757} \end{bmatrix}$$

which comes from the (non-unit) rational quaternion

$$q_{rat} = \begin{bmatrix} \frac{187}{250} & \frac{327}{500} & \frac{27}{250} & \frac{3}{250} \end{bmatrix}$$

Reduced lexicographic groebner basis of IKT equations

IKT equations:

$$\prod_{i=4}^{6} M_i^{i-1}(c_i, s_i, d_i, a_i, \alpha_i) - \left(\prod_{i=1}^{3} M_i^{i-1}(c_i, s_i, d_i, a_i, \alpha_i)\right)^{-1} \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} = \mathbf{O},$$

$$c_i{}^2 + s_i{}^2 = 1, \quad i = 1, \ldots, 6$$

Let's order the variables as follows:

$$c_1 > s_1 > \cdots > c_6 > s_6$$

Then the reduced lexicographic groebner basis of IKT equations looks like:

$$s_6{}^{16} + a_{1,15} \cdot s_6{}^{15} + a_{1,14} \cdot s_6{}^{14} + \cdots + a_{1,1} \cdot s_6 + a_{1,0}$$

$$c_6 + a_{2,15} \cdot s_6{}^{15} + a_{2,14} \cdot s_6{}^{14} + \cdots + a_{2,1} \cdot s_6 + a_{2,0}$$

$$\vdots$$

$$c_1 + a_{12,15} \cdot s_6{}^{15} + a_{12,14} \cdot s_6{}^{14} + \cdots + a_{12,1} \cdot s_6 + a_{12,0}$$