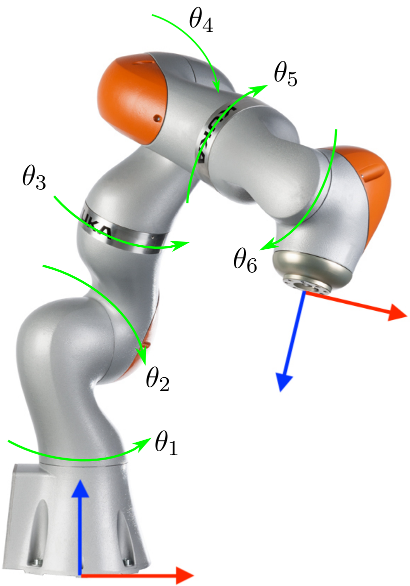
# Inverse Kinematic Task (IKT)



# Mathematical Formulation of IKT

$$M = M_1^0 M_2^1 M_3^2 M_4^3 M_5^4 M_6^5$$

$$\underbrace{\begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}}_{\text{pose of the end effector}} = \prod_{i=1}^{6} M_i^{i-1}(\theta_i + \underbrace{\theta_{i_\text{offset}}, d_i, a_i, \alpha_i}_{\text{DH parameters}})$$

$$\begin{bmatrix} f_1 & \dots & f_4 \\ \vdots & \ddots & \vdots \\ f_9 & \dots & f_{12} \\ 0 & \dots & 0 \end{bmatrix} = \prod_{i=1}^{6} M_i^{i-1}(\theta_i + \theta_{i_\text{offset}}, d_i, a_i, \alpha_i) - \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} = \mathbf{O}$$

We can solve $\mathbf{F}(\theta_1, \dots, \theta_6, \mathbf{R}, \mathbf{t}) = \mathbf{F}(\boldsymbol{\theta}, \mathbf{p}) = \mathbf{0}$ either **numerically** or **symbolically**.

# Numerical methods

1. Newthon's method

2. Homotopy continuation (predictor + Newthon's method)

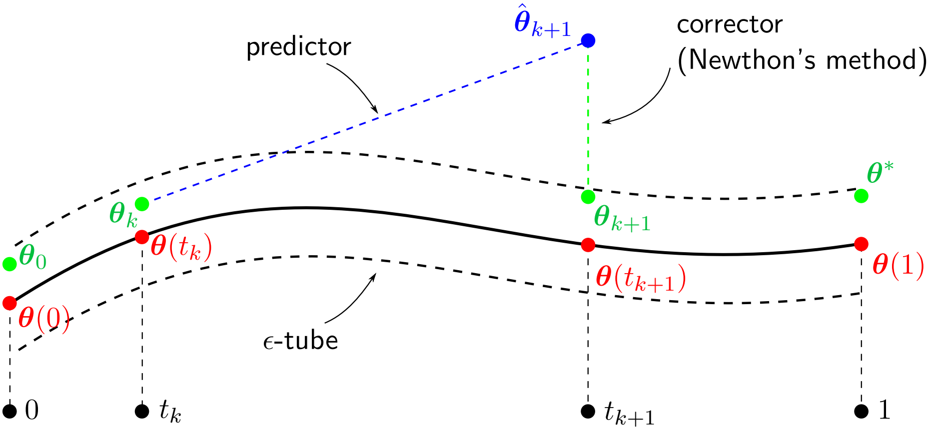
# Newthon's method

Denote by $\boldsymbol{\theta}^*$ one of the solutions to $\mathbf{F}(\boldsymbol{\theta}) = \mathbf{0}$ and by $\mathbf{J}_\mathbf{F}$ the Jacobian $\frac{\partial \mathbf{F}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \in M^{12\times 6}$.
Two basic operations of the Newthon's method are:

$$\boldsymbol{\theta}_0 = \text{something close to } \boldsymbol{\theta}^* \quad \text{(initialization)}$$

$$\boldsymbol{\theta}_{i+1} = \boldsymbol{\theta}_i - \mathbf{J}_\mathbf{F}^+(\boldsymbol{\theta}_i)\mathbf{F}(\boldsymbol{\theta}_i) \quad \text{(improve the previous guess)}$$

# Homotopy continuation




$$\mathbf{H}(\boldsymbol{\theta}, t) = (1-t)\mathbf{F}(\boldsymbol{\theta}, \mathbf{p}_1) + t\,\mathbf{F}(\boldsymbol{\theta}, \mathbf{p}_2)$$

$$\hat{\boldsymbol{\theta}}_{k+1} = \boldsymbol{\theta}_k - \left(\frac{\partial \mathbf{H}}{\partial \boldsymbol{\theta}}(\boldsymbol{\theta}_k, t_k)\right)^+ \frac{\partial \mathbf{H}}{\partial t}(\boldsymbol{\theta}_k, t_k)(t_{k+1} - t_k)$$

# Symbolic method

- Equations $\mathbf{F}(\boldsymbol{\theta})$ are polynomial in $\cos\theta_i$ and $\sin\theta_i$
- New variables: $c_i = \cos\theta_i,\ s_i = \sin\theta_i$
- New **polynomial** equations:

$$\mathbf{F}(c_1, s_1, \ldots, c_6, s_6) \text{ together with } c_i^2 + s_i^2 = 1$$

# Groebner bases (symbolic method)

System of polynomial equations

↓ Buchberger algorithm, F4, F5 (based on Polynomial Division)

Groebner basis (lexicographic ordering)

↓ Solving univariate polynomials + Backsubstitution

Solutions

# Lexicographic monomial ordering

1. Order variables (alphabetical order), i.e.

$$x > y > z$$

2. Every monomial can be written now as $\mathbf{x}^{\boldsymbol{\alpha}}$ with $\mathbf{x} = (x, y, z)$, e.g.

$$\mathbf{x}^{\boldsymbol{\alpha}} = y^2 x^3 z \Rightarrow \boldsymbol{\alpha} = (3, 2, 1)$$

3. We say that

$$\mathbf{x}^{\boldsymbol{\alpha}_1} \geq_{lex} \mathbf{x}^{\boldsymbol{\alpha}_2}$$

   if the left-most nonzero element of $\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2$ is positive or $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2$.

4. For example,

$$x^3 y^3 z \geq_{lex} x^3 y^2 z^{10}, \text{ since } \boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2 = (3, 3, 1) - (3, 2, 10) = (0, 1, -9)$$

# Polynomial Division Algorithm

**Algorithm 1:** Polynomial Division Algorithm

**Input:** $f, F = (f_1, \ldots, f_s), \geq$ (monomial ordering)

**Output:** $(q_1, \ldots, q_s), r$ such that $f = \sum_{i=1}^{s} q_i f_i + r$, $\mathrm{LT}_{\geq}(r)$ is not divisible by any of $\mathrm{LT}_{\geq}(f_i)$ or $r = 0$

$q_1 \leftarrow \ldots \leftarrow q_s \leftarrow r \leftarrow 0$  
$p \leftarrow f$  
**while** $p \neq 0$ **do**  
    $i \leftarrow 1$  
    divisionoccured $\leftarrow FALSE$  
    **while** $i \leq s$ **and** divisionoccured $= FALSE$ **do**  
        **if** $\mathrm{LT}_{\geq}(f_i)$ divides $\mathrm{LT}_{\geq}(p)$ **then**  
            $q_i \leftarrow q_i + \frac{\mathrm{LT}_{\geq}(p)}{\mathrm{LT}_{\geq}(f_i)}$  
            $p \leftarrow p - \frac{\mathrm{LT}_{\geq}(p)}{\mathrm{LT}_{\geq}(f_i)} f_i$  
            divisionoccured $\leftarrow TRUE$  
        **else**  
            $i \leftarrow i + 1$  
    **if** divisionoccured $= FALSE$ **then**  
        $r \leftarrow r + \mathrm{LT}_{\geq}(p)$  
        $p \leftarrow p - \mathrm{LT}_{\geq}(p)$  
**return** $(q_1, \ldots, q_s), r$