***Logical reasoning and programming*, lab session 7**

(November 1, 2021)

7.1 Decide whether for any formula φ holds:

(a) $\varphi \equiv \forall\varphi$,
(b) $\varphi \equiv \exists\varphi$,
(c) $\models \varphi$ iff $\models \forall\varphi$,
(d) $\models \varphi$ iff $\models \exists\varphi$,

where $\forall\varphi$ ($\exists\varphi$) is the universal (existential) closure of φ. If not, does at least one implication hold?

7.2 Show that for any set of formulae Γ and a formula φ holds if $\Gamma \models \varphi$, then $\forall\Gamma \models \varphi$, where $\forall\Gamma = \{\, \forall\psi \colon \psi \in \Gamma \,\}$. Does the opposite direction hold?

7.3 Does it hold $\Gamma \models \varphi$ iff $\forall\Gamma \models \forall\varphi$?

7.4 Find a set of formulae Γ and a formula φ such that $\Gamma \models \varphi$ and $\Gamma \models \neg\varphi$.

7.5 Produce equivalent formulae in prenex form:

(a) $\forall X(p(X) \to \forall Y(q(X,Y) \to \neg\forall Z r(Y,Z)))$,
(b) $\exists X p(X,Y) \to (q(X) \to \neg\forall Z p(X,Z))$,
(c) $\exists X p(X,Y) \to (q(X) \to \neg\exists Z p(X,Z))$,
(d) $p(X,Y) \to \exists Y(q(Y) \to (\exists X q(X) \to r(Y)))$,
(e) $\forall Y p(Y) \to (\forall X q(X) \to r(Z))$.

7.6 In **7.5** you could obtain in some cases various prefixes; the order of quantifiers can be different. Are all these variants correct?

7.7 Can we produce a formula equivalent to **7.5**e with just one quantifier?

7.8 Produce Skolemized formulae equisatisfiable with those in **7.5**. Try to produce as simple as possible Skolem functions.

7.9 Skolemize the following formula

$$\forall X(p(a) \vee \exists Y(q(Y) \wedge \forall Z(p(Y,Z) \vee \exists U q(X,Y,U)))) \vee \exists W q(a,W).$$

Why is it possible in this particular case to do that without producing an equivalent formula in prenex form?

7.10 Unify the following pairs of formulae:

(a) $\{p(X,Y) \doteq p(Y,f(Z))\}$,
(b) $\{p(a,Y,f(Y)) \doteq p(Z,Z,U)\}$,
(c) $\{p(X,g(X)) \doteq p(Y,Y)\}$,
(d) $\{p(X,g(X),Y) \doteq p(Z,U,g(U))\}$,
(e) $\{p(g(X),Y) \doteq p(Y,Y), p(Y,Y) \doteq p(U,f(W))\}$.

Note: You can check your results in SWISH using `unify_with_occurs_check/2`.

7.11 What is the size of the maximal term that is produced when you try to unify

$$\{f(g(X_1, X_1), g(X_2, X_2), \ldots, g(X_{n-1}, X_{n-1})) \doteq f(X_2, X_3, \ldots, X_n\}.$$

7.12 The compactness theorem in First-Order Logic says that a set of sentences has a model iff every finite subset of it has a model. Use this theorem to show that the transitive closure is not definable in FOL.

7.13 Show that the resolution rule is correct.

7.14 Derive the empty clause $\Box$ using the resolution calculus from:

(a) $\{\{\neg p(X), \neg p(f(X))\}, \{p(f(X)), p(X)\}, \{\neg p(X), p(f(X))\}\}$

(b) $\{\{\neg p(X, a), \neg p(X, Y), \neg p(Y, X)\}, \{p(X, f(X)), p(X, a)\}, \{p(f(X), X), p(X, a)\}\}$

7.15 Prove using the resolution calculus that from

$$\forall X \forall Y (p(X, Y) \rightarrow p(Y, X))$$
$$\forall X \forall Y \forall Z ((p(X, Y) \wedge p(Y, Z)) \rightarrow p(X, Z))$$
$$\forall X \exists Y p(X, Y)$$

follows $\forall X p(X, X)$.

7.16 List all the possible applications of the factoring rule on the clause

$$\{p(X, f(Y), Z), \neg s(Z, T), p(T, T, g(a)), p(f(b), S, g(W)), \neg s(c, d)\}.$$

If it is possible to use the factoring rule several times, then produce even these results.

7.17 Check PyRes; simple resolution-based theorem provers for first-order logic. You can find proofs for the previous examples using them.