Auctions 2

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Efficiency of Single-Item Auctions?

Efficiency in single-item auctions: the item allocated to the agent who values it the most.

With independent private values (**IPV**):

Auction	Efficient
English (without reserve price)	yes
Japanese	yes
Dutch	no
Sealed bid second price	yes
Sealed bid first price	no

Note: Efficiency (often) lost in the **correlated** value setting.

Optimal Auctions

Optimal Auction Design

The seller's problem is to **design an auction mechanism** which has a Nash equilibrium giving him/her the **highest possible expected utility**.

- assuming individual rationality

Second-prize sealed bid auction **does not maximize** expected revenue → not the best choice if profit maximization is important (in the short term).

Designing an Optimum Auction

We assume the **IPV setting** and **risk-neutral bidders**.

Each bidder i's valuation is drawn from some **strictly increasing** cumulative density function $F_i(v)$, having probability density function $f_i(v)$ that is continuous and bounded below.

- Allow $F_i(v) \neq F_j(v)$: **asymmetric** valuations

The **risk neutral** seller knows each F_j and has **zero value** for the object.

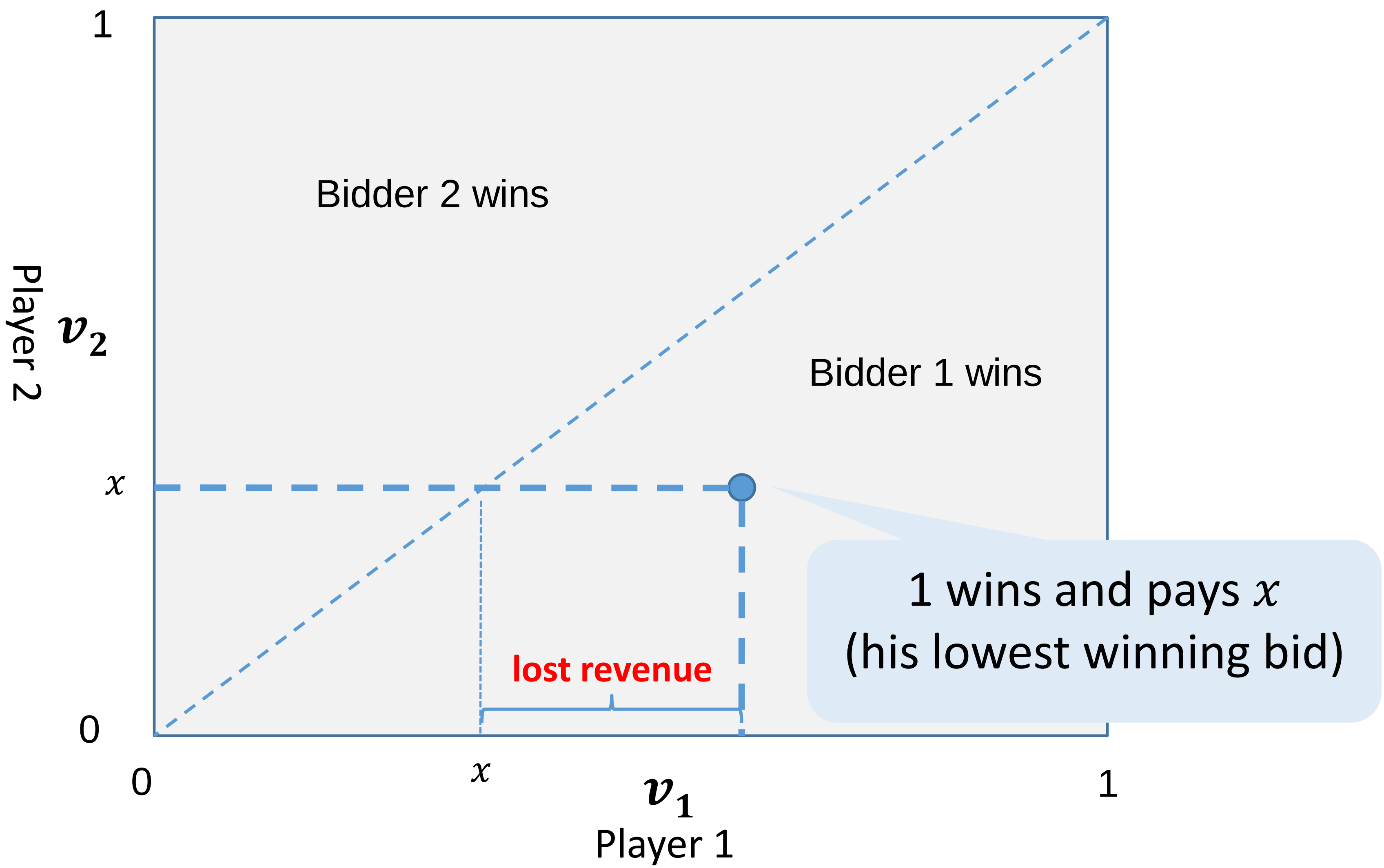
The auction that maximizes the **seller's expected revenue** subject to **individual rationality** and **Bayesian incentive-compatibility** for the buyers is an **optimal auction**.

Example

2 bidders, v_i **uniformly** distributed on $[0,1]$.

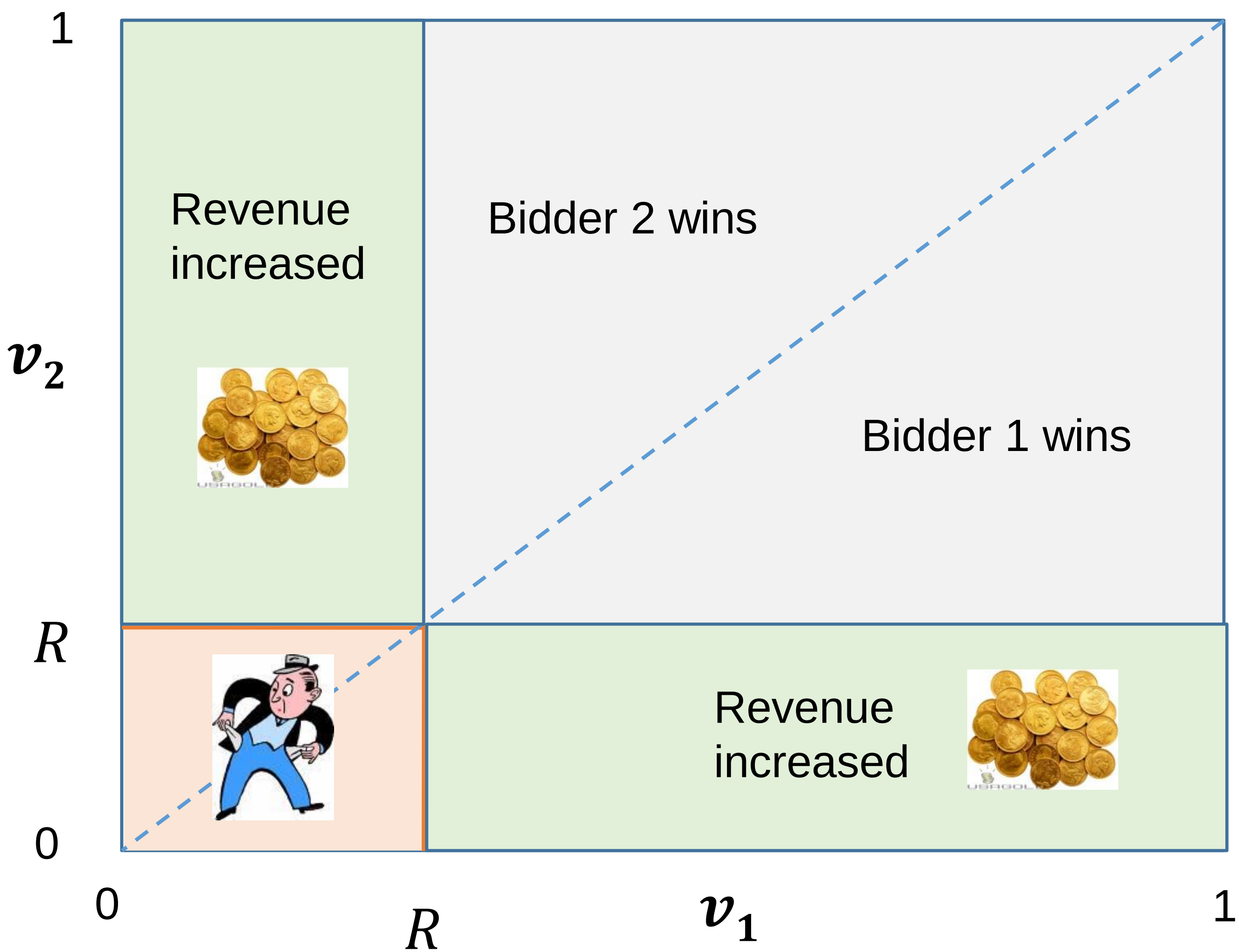
Second-price sealed bid auction.

Outcome without reserve price



Outcome with reserve price

Some reserve price **improves revenue**.



Outcome with reserve price

Bidding true value is still the dominant strategy, so:

1. [Both bides below R]: **No sale.**
   This happens with probability R^2 and then **`revenue`**=0
2. [One bid above the reserve and the other below]: Sale at **reserve price *R***
   This happens with probability $2(1-R)R$ and the **`revenue`**= R
3. [Both bids above the reserve]: Sale at the **second highest bid**.
   This happens with probability $(1-R)^2$ and the
   `revenue`= $E[\min v_i \,|\, \min v_i \geq R] = \frac{1+2R}{3}$

Expected **`revenue`** $= 2(1-R)R^2 + (1-R)^2 \frac{1+2R}{3}$

$$= \frac{1+3R^2-4R^3}{3}$$

Maximizing: $0 = 2R - 4R^2$, i.e., $R = \frac{1}{2}$

Outcome with reserve price

Reserve price of $1/2$: $\mathbf{revenue} = 5/12$

Reserve price of 0: $\mathbf{revenue} = 1/3 = 4/12$

Tradeoffs:

- **Lose the sale** when both bids below $1/2$: but low revenue then in any case and probability $1/4$ of happening.
- **Increase the sale price** when one bidder has low valuation and the other high: happens with probability $1/2$.

Setting a reserve price is like **adding another bidder**: it increases competition in the auction.

Optimal Single Item Auction

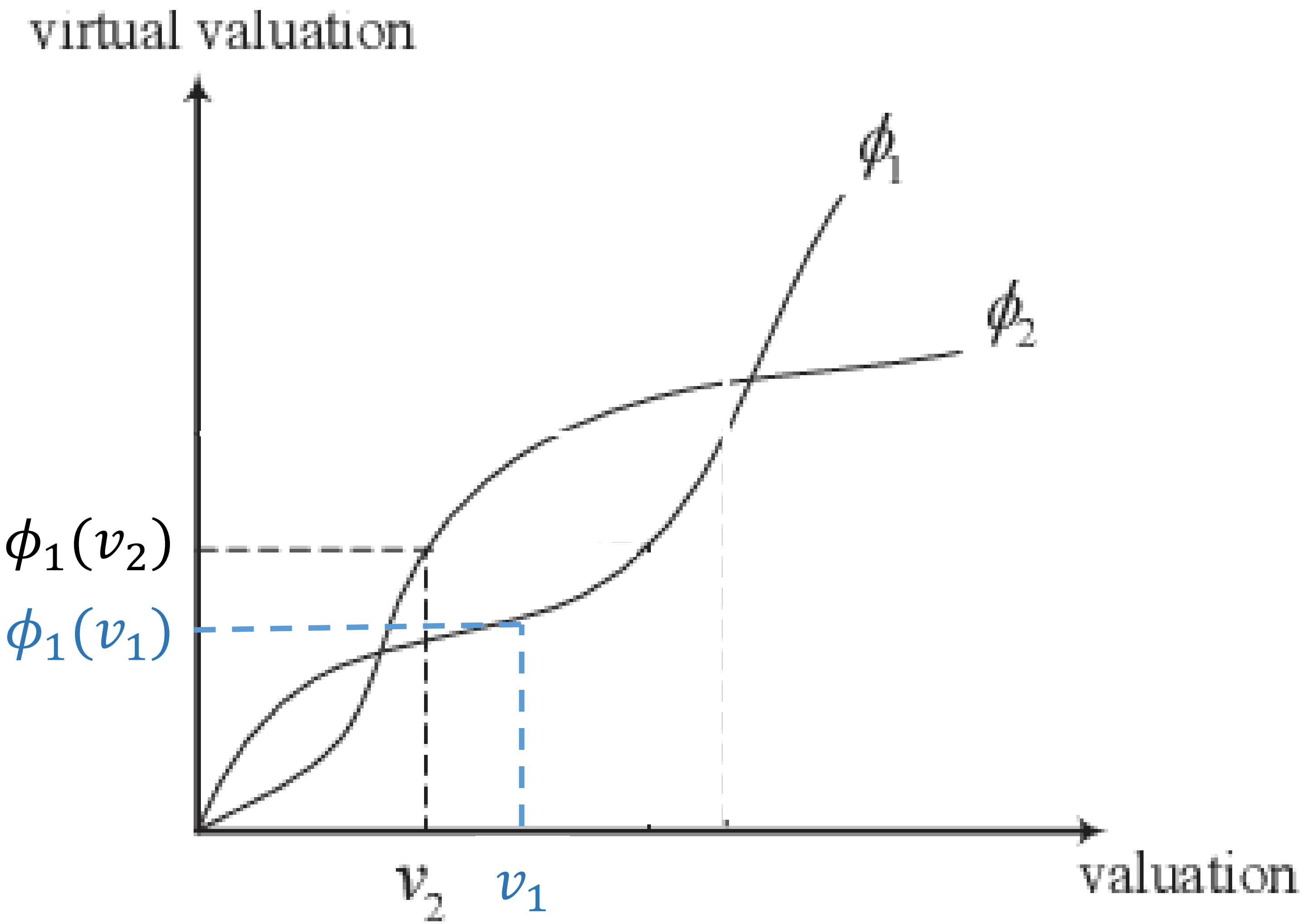
Definition (Virtual valuations)

Consider an **IPV setting** where bidders are **risk neutral** and each bidder i's valuation is drawn from some **strictly increasing** cumulative density function $F_i(v)$, having probability density function $f_i(v)$. We then define:
where

- Bidder i's **virtual valuation** is $\psi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$
- Bidder i's **bidder-specific reserve price** r_i^* is the value for which $\psi_i(r_i^*) = 0$

Example: uniform distribution over $[0,1]$: $\psi(v) = 2v - 1$

Example virtual valuation functions



Optimal Single Item Auction

Theorem (Optimal Single-item Auction)

The **optimal (single-good) auction** is a sealed-bid auction in which every agent is asked to **declare his valuation**. The good is sold to the agent $\boldsymbol{i = \mathrm{argmax_i} \psi_i(\hat{v}_i)}$**,** as long as $\hat{v}_i > r_i^*$.
If the good is sold, the winning agent i is charged the smallest valuation that it could have declared while still remaining the winner:

$$\inf\{v_i^* : \psi_i(v_i^*) \geq 0 \wedge \forall j \neq i, \psi_i(v_i^*) \geq \psi_j(\hat{v}_j)\}$$

Can be understood as a second-price auction with a reserve price, held **in virtual valuation space** rather than in the space of actual valuations.

Remains **dominant-strategy truthful.**

Second-Price Auction with Reservation Price

Symmetric case: second-price auction with reserve price r^* satisfying: $\psi(r^*) = r^* - \frac{1-F(r^*)}{f(r^*)} = 0$

- **Truthful** mechanism when $\psi(v)$ is non-decreasing.
- Uniform distribution over $[0, p]$: optimum reserve price $= p/2$.

Second-price sealed bid auction with Reserve Price is **not efficient**!

Second-Price Auction with Reservation Price

Why does this increase revenue?

- Reservation prices are like **competitors**: increase the payments of winning bidders.
- The virtual valuation can increase the impact of weak bidders' bids, making the **more competitive**.
- Bidders with higher expected valuations bid **more aggressively**.

Optimal Auctions: Remarks

For **optimal revenue** one needs to **sacrifice** some **efficiency**.

Optimal auctions are not **detail-free:**

- they require the seller to incorporate information about the bidders' valuation distributions into the mechanism
- → rarely used in practice

Theorem (Bulow and Klemperer): *revenue* of an efficiency-maximizing auction with *k+1* bidder is at least as high as that of the revenue-maximizing one with *k* bidders.

➔ better to spend energy on attracting more bidders

Combinatorial Auctions

Auctions for **bundles of goods.**

Let $\mathcal{G} = \{g_1, \dots, g_n\}$ be a set of items (goods) to be auctioned

A **valuation function** $v_i: 2^{\mathcal{G}} \mapsto \mathbb{R}$ indicates how much a bundle $G \subseteq \mathcal{G}$ is worth to agent i.

We typically assume the following properties:

- **normalization**: $v(\emptyset) = 0$
- **free disposal**: $G_1 \subseteq G_2$ implies $v(G_1) \leq v(G_2)$

Non-Additive Valuations

Combinatorial auctions are interesting when the valuation function is **not additive.**

Two main types on non-additivity.

Substitutability	Complementarity
The valuation function v exhibits **substitutability** if there exist two sets of goods $G_1, G_2 \subseteq G$ such that $G_1 \cap G_2 = \emptyset$ and $v(G_1 \cup G_2) < v(G_1) + v(G_2)$. Then this condition holds, we say that the valuation function v is **subadditive**.	The valuation function v exhibits **complementarity** if there exist two sets of goods $G_1, G_2 \subseteq G$ such that $G_1 \cap G_2 = \emptyset$ and $v(G_1 \cup G_2) > v(G_1) + v(G_2)$. Then this condition holds, we say that the valuation function v is **superadditive**.
Ex: Two different brands of TVs.	Ex: Left and right shoe.

How to Sell Goods with Non-Additive Valuations?

1. Ignore valuations dependencies and sell sequentially via a sequence of **independent single-item** auctions.

   → **Exposure problem**: A bidder may bid aggressively for a set of goods in the hope of winning a bundle but, only succeed in winning a subset (a thus paying to much)

2. Run separate but **connected single-item** auctions **simultaneously**.
   - a bidder bids in one auction he has a reasonably good indication of what is transpiring in the other auctions of interest.

3. **Combinatorial auction**: bid directly on a **bundle of goods.,**

Allocation in Combinatorial Auction

Allocation is a list of sets $G_1, \dots, G_n \subseteq \mathcal{G}$, one for each agent i such that $G_i \cap G_j = \emptyset$ for all $i \neq j$ (i.e. not good allocated to more than one agent)

Which way to choose an allocation for a combinatorial auction?

→ The simples is to maximize **social welfare (efficient allocation)**:

$$U(G_1, \dots, G_n, v_1, \dots, v_n) = \sum_{i=1}^{n} v_i(G_i)$$

Simple Combinatorial Auction Mechanism

The mechanism determines the **social welfare maximizing allocation** and then **charges** the winners their **bid** (for the bundle they have won), i.e., $\rho_i = \hat{v}_i$.

Example:

Bidder 1	Bidder 2	Bidder 3
$v_1(x, y) = 100$	$v_2(x) = 75$	$v_3(y) = 40$
$v_1(x) = v_1(y) = 0$	$v_2(x, y) = v_2(y) = 0$	$v_3(x, y) = v_3(x) = 0$

Is this incentive-compatible? **No.**

VCG auction

A **Vickrey–Clarke–Groves (VCG) auction** is a type of sealed-bid auction of multiple items. Bidders submit bids that report their valuations for the items, without knowing the bids of the other bidders. The auction system assigns the items in a socially optimal manner: it charges each individual the harm they cause to other bidders.[1]

Vickrey–Clarke–Groves (VCG) auction, an analogy to **second-price** sealed bid single-unit auctions, exists for the combinatorial setting and it **is dominant-strategy truthful** and **efficient**.

VCG example

Suppose two apples are being auctioned among three bidders.

- Bidder A wants one apple and is willing to pay $5 for that apple.
- Bidder B wants one apple and is willing to pay $2 for it.
- Bidder C wants two apples and is willing to pay $6 to have both of them but is uninterested in buying only one without the other.

First, the outcome of the auction is determined by maximizing social welfare:

- the apples go to bidder A and bidder B, since their combined bid of $5 + $2 = $7 is greater than the bid for two apples by bidder C who is willing to pay only $6.
- Thus, after the auction, the value achieved by bidder A is $5, by bidder B is $2, and by bidder C is $0 (since bidder C gets nothing).

Payment of bidder **A**:

- an auction that excludes bidder A, the social-welfare maximizing outcome would assign both apples to bidder C for a total social value of $6.
- the total social value of the original auction *excluding A's value* is computed as $7 - $5 = $2.
- Finally, subtract the second value from the first value. Thus, the payment required of A is $6 − $2 = $4.

Payment of bidder **B**:

- the best outcome for an auction that excludes bidder B assigns both apples to bidder C for $6.
- The total social value of the original auction *minus B's portion* is $5. Thus, the payment required of B is $6 − $5 = $1.

Finally, the payment for bidder C is (($5 + $2) − ($5 + $2)) = $0.

After the auction, A is $1 better off than before (paying $4 to gain $5 of utility), B is $1 better off than before (paying $1 to gain $2 of utility), and C is neutral (having not won anything).

Winner Determination Problem

Definition

The **winner determination problem** for a combinatorial auctions, given the agents' declared valuations $\hat{v}_i$ is to find the social-**welfare-maximizing allocation** of goods to agents. This problem can be expressed as the following integer program

$$\text{maximize} \sum_{i \in N} \sum_{Z \subseteq \mathcal{Z}} \hat{v}_i(Z) x_{Z,i}$$

$$\text{subject to} \sum_{Z, j \in Z} \sum_{i \in N} x_{Z,i} \leq 1 \quad \forall j \in \mathcal{Z}$$

$$\sum_{Z \subseteq \mathcal{Z}} x_{Z,i} \leq 1 \qquad \forall i \in N$$

$$x_{Z,i} = \{0,1\} \qquad \forall Z \subseteq \mathcal{Z}, i \in N$$

Complexity of the Winner Determination Problem

Equivalent to a **set packing problem** (SSP) which is known to be **NP-complete**.

Worse: SSP cannot be **approximated uniformly** to a fixed constant.

Two possible solutions:

- **Limit** to instance where polynomial-time solutions exist.
- **Heuristic methods** that drop the *guarantee* of polynomial runtime, optimality or both.

Bid Representation

The problem: How to **encode** the bid (i.e. the valuation function) in a **succinct** (polynomial-size) form?

Expressivity vs. conciseness.

Atomic bids

Bidding for just **one particular subset** of goods.

An **atomic bid** is a pair (S, p) indicating the agent is willing to pay **price $\boldsymbol{p}$** for the subset of **goods $\boldsymbol{S}$**.

Example: The agents wants to pay \$100 for a bundle of a TV and a gaming console.

Very **limited expressive power**: not even the basic **additive valuation** function can be represented.

OR bids

More expressive than atomic bids.

OR bid is a **disjunction** of atomic bids

$$(S_1, p_1) \lor (S_2, p_2) \lor \cdots \lor (S_k, p_k)$$

that indicates that the agent is willing to pay a price of p_1 for the subset of goods S_1, or a price of p_2 for the subset of goods S2, etc.

We interpret OR as an **operator for combining valuation functions**. Let V be the space of possible valuation functions, and $v_1, v_2 \in V$ be arbitrary valuation functions. Then we have that

$$(v_1 \lor v_2)(S) = \max_{R,T \subseteq S, R \cap T = \emptyset} (v_1(R) + v_2(T)).$$

OR bids expressivity

OR bid **can** express **additive valuations** but still quite limited.

Theorem: OR bids can express all valuation functions that exhibit **no substitutability**, and only these.

Example:

- Let's have two goods x and y and a valuation function $v(x) = v(y) = 10$ and $v(x, y) = 15$.
- This valuation function cannot be expressed an OR bid because $\max(v\ (x) + v(y), v(x, y)) = 20$, i.e., the interpretation of the OR bid would ascribe the valuation of 20 to the (x, y) bundle.

XOR bids

XOR bids are more powerful.

XOR bid is an **exclusive OR** of atomic bids $(S_1, p_1) \oplus (S_2, p_2) \oplus \cdots \oplus (S_k, p_k)$ that indicates that the agent is willing to accept one but no more than one of the atomic bids.

The XOR operator is defined on the space of valuation functions. Let V be the space of possible valuation functions, and $v_1, v_2 \in V$ be arbitrary valuation functions. The we have that

$$(v_1 \oplus v_2)(S) = \max(v_1(S), v_2(S))$$

Example: $(\{\mathrm{TV}, \mathrm{DVD}\}, 100) \oplus (\{\mathrm{TV}, \mathrm{Dish}\}, 150)$.

XOR bids expressivity

Theorem: XOR bids can represent **all** possible valuation functions.

But: Not every valuation function can be represented efficiently by XOR bids.

In fact, the simple **additive** valuations can be represented by short OR bids but require XOR bids of **exponential size**.

The OR* bidding language

We can simulate the effect of an XOR by allowing bids to include **dummy** (or **phantom**) items.

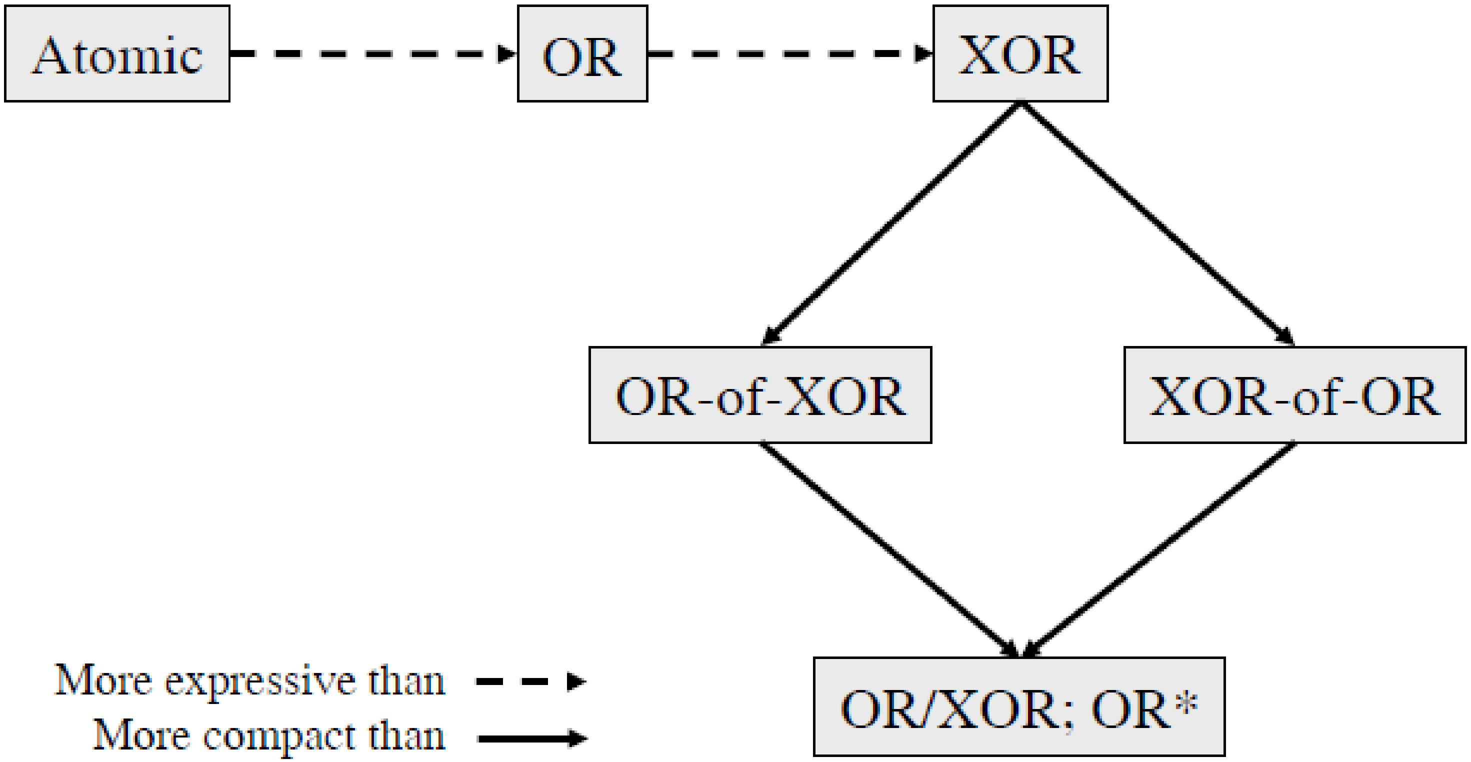
Definition (OR* bid) Given a set of dummy items G_i for each agent $i \in N$, an OR* bid is a **disjunction of atomic bids** $(S_1, p_1) \vee (S_2, p_2) \vee \cdots \vee (S_k, p_k)$, where for each $l = 1, \ldots, k$, the agent is willing to pay a price of p_l for the set of items $S_l \subseteq G \cup G_i$.

Example:

$$(\{TV, D\}, 100) \vee (\{DVD, D\}, 100) \vee (\{TV, DVD, D\}, 150)$$

OR* **can express all bids** and is **more succinct** than OR, XOR languages and their combinations.

Relationships between Bid Languages



However, **interpretation complexity** can be non-polynomial.

Auctions Summary

Auctions are mechanisms for **allocating scarce resource** among **self-interested agent**

Mechanism-design and game-theoretic perspective

Many auction mechanisms: English, Dutch, Japanese, First-price sealed bid, Second-price sealed bid

Desirable properties: truthfulness, efficiency, optimality, ...

Rapidly expanding list of **applications** worth billions of dollars

Reading:

- [Shoham] – Chapter 11