# Multiagent Resource Allocation

Michal Jakob
Artificial Intelligence Center,
Dept. of Computer Science and Engineering,
FEE, Czech Technical University

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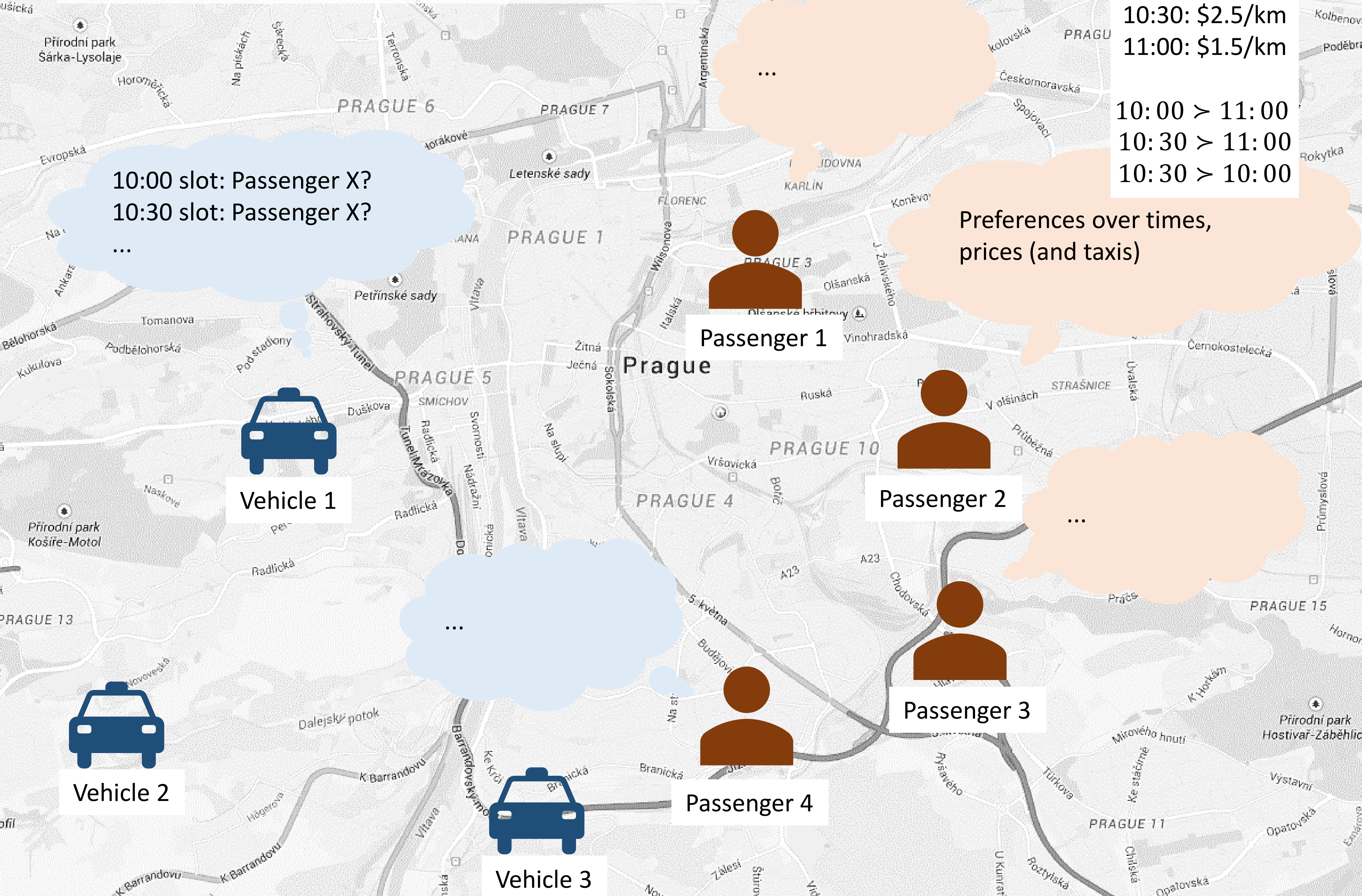
# Motivating Example

Taxi is a limited resources


*Who should get the taxi and when (and possibly at which price)?*

# Motivating Example



# Multiagent Resource Allocation (MARA)

# What is Multiagent Resource Allocation?

*Multiagent Resource Allocation (MARA) is the process of distributing a number of items amongst a number of agents.*

- **What** kind of items (resources) are being distributed?
- **How** are they being distributed?
- **Why** are they being distributed?

# Classification of MARA

1. **Resources** (What)
2. Agent (i.e. individual) **preferences** (Why)
3. Social (i.e. collective) **welfare** (Why)
4. Allocation **mechanism** (How)

Link to **social choice**: allocations are the alternatives over which agents express their preferences.

Link to **game theory**: allocation mechanisms are games (that needs to be designed and for which strategies can be studied).

# Types of Resources

Different **types** of resources may require different resource allocation **techniques**.

- **Continuous** vs. **Discrete**
  Continuous resource can be arbitrarily divided.
- **Divisible** vs. **Indivisible**
  Discrete resources indivisible; continuous can be treated either way.
- **Sharable** vs. **Non-Sharable**
  Sharable can be assigned multiple times: e.g. a path in a network.
- **Static** vs. **Non-Static**
  static = properties do not change; non-static = properties do change e.g. perishable goods.
- **Single-Unit** vs. **Multi-Unit**
  One copy vs. multiple copies (ten trucks of the same type).

# Resources vs. Tasks

**Tasks** may be considered resources with **negative** utility (cost).

**Task allocation** may be regarded a multiagent resource allocation problem.

- However, tasks are often coupled with **constraints** regarding their **coherent combination** (timing and ordering).

# Preference Representation

How can we represent agent’s preferences over allocations?

# Preference Representation

Agents may have **preferences** over

- the bundle of resources they receive
- the bundles of resources received by others (**externalities**)

What are suitable **languages** for representing agent **preferences**?

# Notation

Set of **agents** $\mathcal{A} = \{1, \dots, n\}$

Set of **resources** $\mathcal{R}$

Agents have **preferences over allocations** $X \in \mathcal{X}$

**Allocation** $X$ is a *partial* mapping of $\mathcal{R}$ to $\mathcal{A}$
(not all resources need to be allocated)

# Cardinal vs. Ordinal Preferences

A **preference structure** represents an agent's preferences over allocations $X \in \mathcal{X}$.

| Cardinal preferences | Ordinal preferences |
|---|---|
| Cardinal preference structure is a **function** $u: \mathcal{X} \mapsto Val$, where $Val$ is usually a set of numerical values such as $\mathbb{N}$ or $\mathbb{R}$ (and typically non-negative) | Ordinal preference structure is a *reflexive, transitive and connected* **binary *relation*** $\preccurlyeq$ on the set $\mathcal{X} \times \mathcal{X}$ which can used to *compare* allocations. |

If the allocations over which agents must express preferences are *bundles of indivisible resources* from the set $\mathcal{R}$, then we have $\mathcal{X} = 2^{\mathcal{R}}$.

# Example

Hanging a picture with a **frame** (f), a **hammer** (h) and a **nail** (n)

Cardinal

| $X$ | $u(X)$ |
|---|---|
| $\{\}$ | 0 |
| $\{f\}$ | 0 |
| $\{h\}$ | 0 |
| $\{n\}$ | 10 |
| $\{f,h\}$ | 0 |
| $\{f,n\}$ | 20 |
| $\{h,n\}$ | 15 |
| $\{f,h,n\}$ | 50 |

Ordinal

| $\succcurlyeq$ | $\{\}$ | $\{f\}$ | $\{h\}$ | $\{n\}$ | $\{f,h\}$ | $\{f,n\}$ | $\{h,n\}$ | $\{f,h,n\}$ |
|---|---|---|---|---|---|---|---|---|
| $\{\}$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\{f\}$ | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 |
| $\{h\}$ | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 |
| $\{n\}$ | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| $\{f,h\}$ | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 |
| $\{f,n\}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |
| $\{h,n\}$ | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 |
| $\{f,h,n\}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Cardinal can always be translated to ordinal
(but information about the strength of preference lost).
Ordinal cannot be always translated to cardinal.

# Preference Representation Languages

**Expressive power**
Can the chosen language encode all the preference structures we are interested in?

**Succinctness**
Is the representation of (typical) preference structures succinct? Is one language more succinct than the other?

**Complexity**
What is the computational complexity of related decision problems, such as comparing two alternatives?

**Cognitive relevance**
How close is a given language to the way in which humans would express their preferences?

**Elicitation**
How dicult is it to elicit the preferences of an agent so as to represent them in the chosen language?

# Preferences Properties

| | Cardinal | Ordinal |
|---|---|---|
| Intrapersonal comparison | yes | Yes |
| Interpersonal comparison ("Ann likes x more than Bob likes y") | yes | No |
| Preference intensity | yes | No |
| Cognitive relevance | lower | higher |
| Explicit representation | $\mathcal{O}(\lvert\mathcal{X}\rvert)$ | $\mathcal{O}(\lvert\mathcal{X}\rvert^2)$ |

Representation can be an issue ➔ **compact** representations

# Social Welfare

How can we express preferences from the collective perspective?

# Social Welfare

A third parameter in the specification of a MARA problem concerns our goals:

**What kind of allocation do we want to achieve?**

How do we **measure the quality** of an allocation?

→ **Social welfare** is the **metrics** for assessing the **quality** of an **allocation** of resources.

# Efficiency vs. Fairness Example

Allocation problem

- **Agents** $\mathcal{A} = \{Alice, Bob\}$
- **Items** $\mathcal{R} = \{phone, bike, shoes\}$
- **Utility functions:**

| **Resources** | **Alice** $\boldsymbol{u_a(r)}$ | **Bob** $\boldsymbol{u_b(r)}$ |
|---|---|---|
| phone | 20 | 6 |
| bike | 10 | 8 |
| shoes | 10 | 4 |

Additive utilities:

$$u_a(X) = \sum_{r, X(R) = Alice} u_a(r)$$

$$u_b(X) = \sum_{r, X(R) = Bob} u_b(r)$$

*How should we allocate the items?*

# Efficiency and Fairness

Two key components of social welfare.

## Efficiency*

The aspect of efficiency include:

- **Pareto optimality:** There should be no alternative allocation that would be better for some and not worse for any of the other agents that the chosen allocation.
- **Utilitarism** (for cardinal preferences)**:** the sum of all utilities should be as high as possible.

## Fairness

The aspect of fairness include:

- **Envy-freeness:** No agent should prefer to take the bundle allocated to one of its peers rather than keeping their own.
- **Egalitarianism** (for cardinal preferences)**:** The agent that is going to be worst off should be as well off as possible.

*not in the computational sense

# Utilitarian Social Welfare

**Utilitarian Social Welfare**

The **utilitarian** social welfare function (also called collective utility function) $sw_u$ is defined as the sum of individual utilities:

$$sw_u(X) = \sum_{i \in \mathcal{A}} u_i(X)$$

Maximizing utilitarian CUF improves **efficiency**.

The utilitarian CUF is **zero-independent**: adding a constant value to your utility function will not affect social welfare judgements.

The value of the utilitarian social welfare function for a given allocation is also termed **social value** (or **collective utility** or **aggregate utility**) of a given allocation.

# Egalitarian Social Welfare

**Egalitarian Social Welfare**

The **egalitarian** social welfare function $sw_e$ is defined as the sum of individual utilities:

$$sw_e(X) = \min_{i \in \mathcal{A}} u_i(X)$$

Maximising this function amounts to improving the situation of the weakest members of society (→ **fairness**).

# Nash Product Social Welfare

**Nash Social Welfare**

The **Nash** social welfare function $sw_e$ is defined as the sum of individual utilities:

$$sw_n(X) = \prod_{i \in \mathcal{A}} u_i(X)$$

This is a useful measure of social welfare as long as all utility functions can be assumed to be **positive**.

Nash CUF favours increases in overall utility, but also inequality-reducing redistributions $(2 \cdot 6 < 4 \cdot 4)$ ➔ **proportional fairness**.

The Nash CUF is **scale independent**: whether a particular agent measures their own utility in euros or dollars does not affect social welfare judgements.

# Efficiency vs. Fairness Example

## Allocation problem

- **Agents** $\mathcal{A} = \{Alice, Bob\}$
- **Items** $\mathcal{R} = \{phone, bike, shoes\}$
- **Utility functions:**

| **Resources** | **Alice** $\boldsymbol{u_a(r)}$ | **Bob** $\boldsymbol{u_b(r)}$ |
|---|---|---|
| phone | 20 | 7 |
| bike | 10 | 5 |
| shoes | 10 | 3 |

Additive utilities:

$$u_a(X) = \sum_{r, X(R)=Alice} u_a(r)$$

$$u_b(X) = \sum_{r, X(R)=Bob} u_b(r)$$

## Allocations

| **Res.** | **Allocations** | | |
|---|---|---|---|
| | **efficient** $\boldsymbol{X_e}$ | **fair** $\boldsymbol{X_f}$ | **Propor.** $\boldsymbol{X_n}$ |
| phone | Alice | Bob | Alice |
| bike | Alice | Bob | Bob |
| shoes | Alice | Alice | Bob |
| $sw_u$ | **40** | 22 | 28 |
| $sw_f$ | 0 | **10** | 8 |
| $sw_n$ | 0 | 120 | **160** |

# Efficiency vs. Fairness Trade-off

# Efficient Allocation

We assume **cardinal preferences**.

Utilitarian welfare function is considered to **measure the efficiency** of an allocation.

An allocation is called **efficient** (also **utilitarian**) if it **maximizes** the sum of utilities of all agents (=**social value**).

We denote the social value of an efficient allocation as $\mathrm{EFFICIENT}(\mathcal{X})$, i.e.,

$$\mathrm{EFFICIENT}(\mathcal{X}) = \sup\{\textstyle\sum_{i\in\mathcal{A}} u_i(X)\,|X \in \mathcal{X}\}$$

# $\alpha$-Fair Allocation

**Constant Elasticity Social Welfare Function**

**Constant Elasticity** Social Welfare Function $sw_\alpha$ with **inequality aversion parameter** $\alpha$ is defined as

$$sw(X, \alpha) = \begin{cases} \sum_{i \in \mathcal{A}} \frac{u_i(X)^{1-\alpha}}{1-\alpha} & \text{for } \alpha \geq 0, \alpha \neq 1 \\ \sum_{i \in \mathcal{A}} \log u_i(X) & \text{for } \alpha = 1 \end{cases}$$

$\alpha = 0$: Utilitarian SWF

$\alpha = 1$: Proportional fairness (~Nash SWF)

$\alpha \to \infty$: Egalitarian (Max-min) SWF

# $\alpha$-Fair Allocation

**$\boldsymbol{\alpha}$-fair allocation** $X^*(\alpha)$ is an allocation that maximizes the constant elasticity social welfare function for the corresponding value of $\alpha$, i.e,

$$X^*(\alpha) = \operatorname*{argmax}_{X \in \mathcal{X}} sw(X, \alpha)$$

We denote the social value of the $\alpha$-fair allocation as $\mathbf{FAIR}(\boldsymbol{\mathcal{X}}, \boldsymbol{\alpha})$, i.e.,

$$\text{FAIR}(\mathcal{X}, \alpha) = sw_{u}(X^*(\alpha))$$

# Price of Fairness

Quantifies the **loss of efficiency** due to the requirement for fairness.

**Price of Fairness**

$$\text{POF}(\mathcal{X}, \alpha) = \frac{\text{EFFICIENT}(\mathcal{X}) - \text{FAIR}(\mathcal{X}, \alpha)}{\text{EFFICIENT}(\mathcal{X})}$$

Price is fairness is always **between zero and one**, and it corresponds to the **percentage efficiency loss** compared to the maximum system efficiency.

Note: $\text{POF}(\mathcal{X}, 0) = 0$

# PoF: Example

Allocation problem

- **Agents** $\mathcal{A} = \{Alice, Bob\}$
- **Items** $\mathcal{R} = \{phone, bike, shoes\}$
- **Utility functions:**

| **Resources** | **Alice** $\boldsymbol{u_a(r)}$ | **Bob** $\boldsymbol{u_b(r)}$ |
|---|---|---|
| phone | 20 | 7 |
| bike | 10 | 5 |
| shoes | 10 | 3 |

$$POF(\mathcal{X}, \infty) = \frac{40 - 22}{40} = \frac{11}{20}$$

$$POF(\mathcal{X}, 1) = \frac{40 - 28}{40} = \frac{3}{10}$$

# Price of Fairness

**Theorem**

Consider a resource allocation problem with $n \geq 2$ agents where all agents have non-negative utilities with the same maximum achievable utility and the set of all feasible utility allocation is convex.

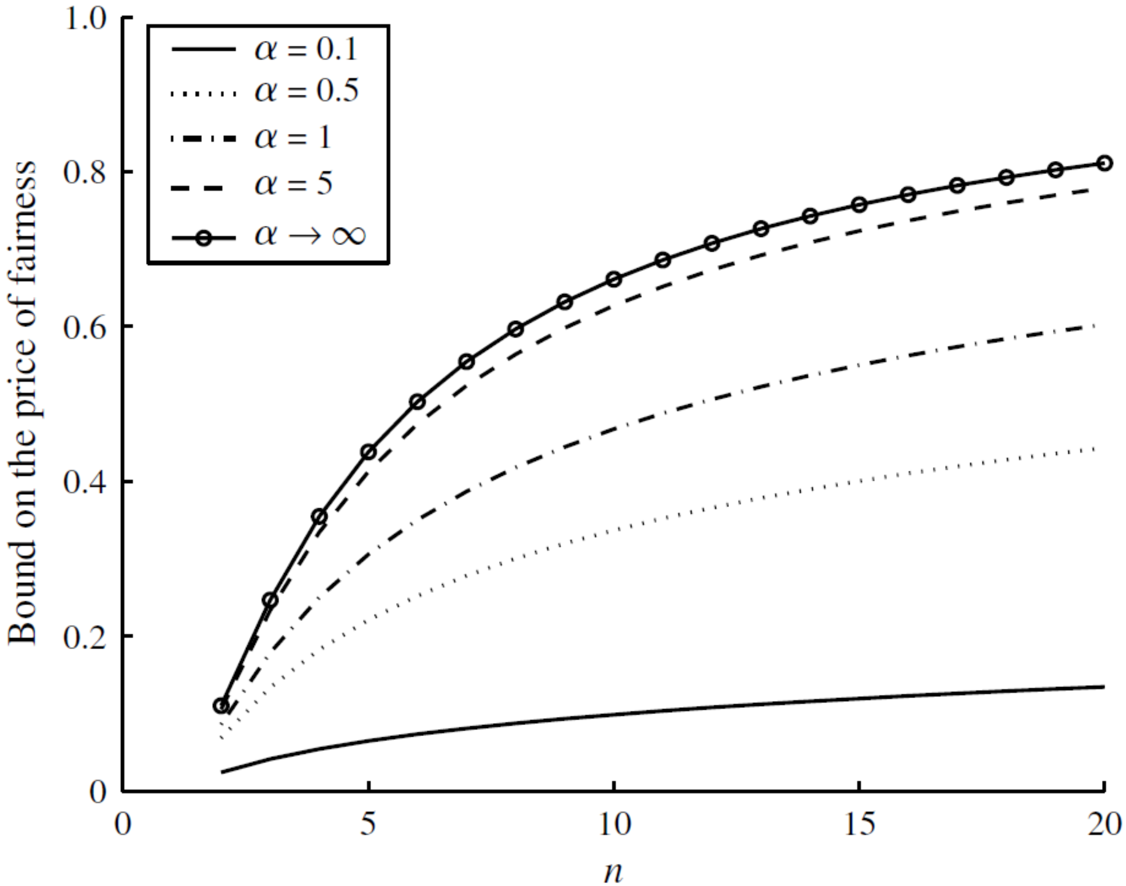
Then for the $\alpha$-fair allocations, $\alpha \geq 0$, **the price of fairness** is bounded by

$$\mathrm{POF}(\mathcal{X}, \alpha) \leq 1 - \Theta(n^{-\frac{\alpha}{1+\alpha}})$$

Generalization to heterogeneous utilities possible

- the price then increases with the ratio between the highest and lowest achievable utility

# Price of Fairness




The worst-case price is **increasing** with the **number of agents** and the **value of $\alpha$**.

Bounds are very strong, **near-tight**.

# Price of Efficiency

Quantifies the **loss of fairness** due to the requirement for efficiency.

We adopt the **minimum utility** egalitarian social welfare function as the fairness metric.

**Price of Efficiency**

$$\mathrm{POE}(\mathcal{X}, \alpha) = \frac{\max_{X \in \mathcal{X}} \min_{i \in \mathcal{A}} u_i(X) - \min_{i \in \mathcal{A}} u_i(X_i^*(\alpha))}{\max_{X \in \mathcal{X}} \min_{i \in \mathcal{A}} u_i(X)}$$

(where $X^*(\alpha)$ is the $\alpha$-fair allocation)

# PoE: Example

## Allocation problem

- **Agents** $\mathcal{A} = \{Alice, Bob\}$
- **Items** $\mathcal{R} = \{phone, bike, shoes\}$
- **Utility functions:**

| Resources | Alice $u_a(r)$ | Bob $u_b(r)$ |
|---|---|---|
| phone | 20 | 7 |
| bike | 10 | 5 |
| shoes | 10 | 3 |

Additive utilities:

$$u_a(X) = \sum_{r, X(R)=Alice} u_a(r)$$

$$u_b(X) = \sum_{r, X(R)=Bob} u_b(r)$$

$$POE(\mathcal{X}, 0) = \frac{10 - 0}{10} = 1$$

$$POE(\mathcal{X}, 1) = \frac{10 - 8}{10} = \frac{1}{5}$$

# Price of Efficiency

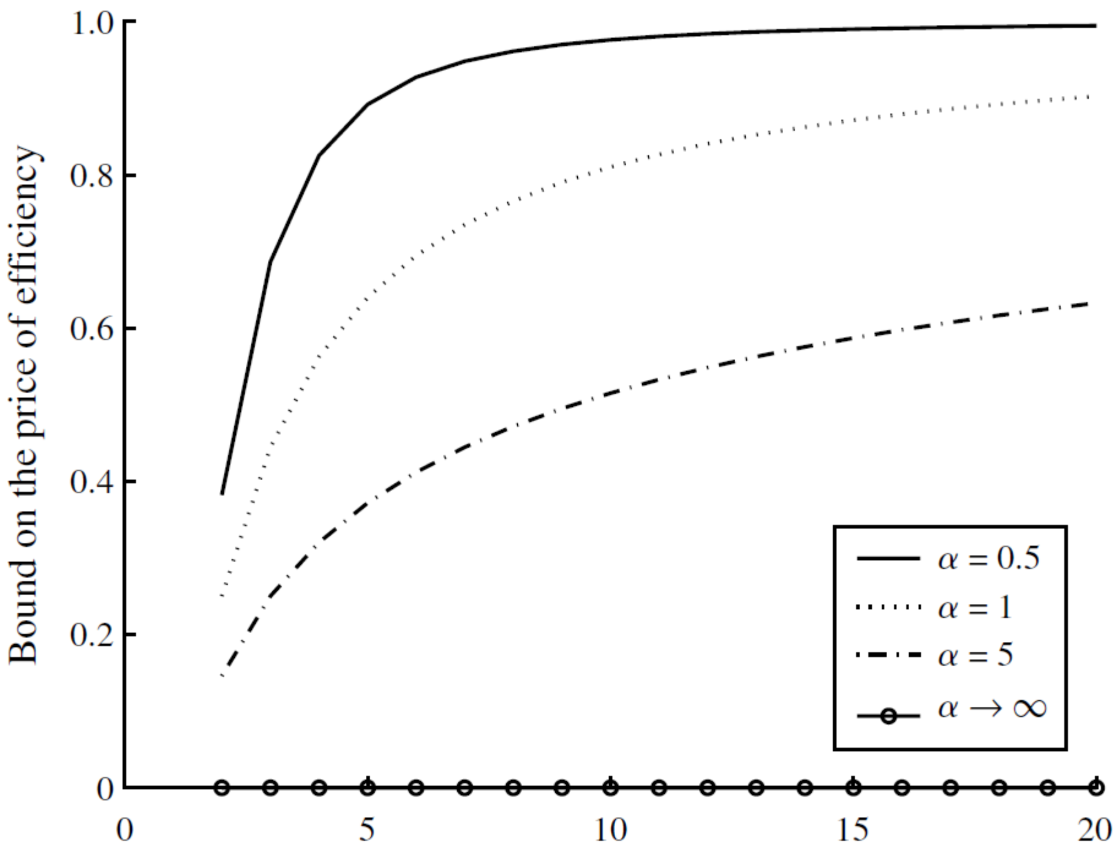
**Theorem**

Consider a resource allocation with $n \geq 2$ agents where all agents have non-negative utilities and the same maximum achievable utility and the set of all feasible utility allocations is convex.

Then for the $\alpha$-fair allocations, $\alpha \geq 0$, the **price of efficiency** is bounded by

$$\mathrm{POE}(\mathcal{X}, \alpha) \leq 1 - \Theta(n^{-\frac{1}{\alpha}})$$

# Price of Efficiency

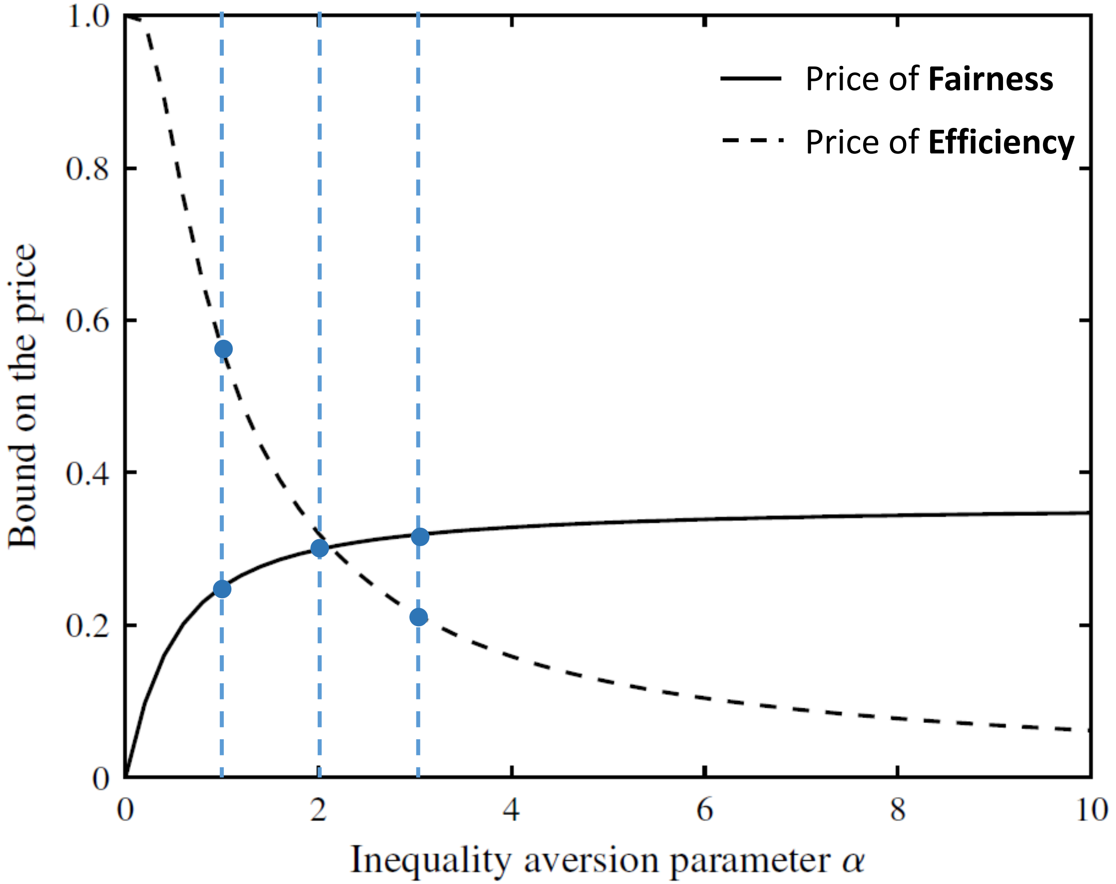



The worst-case price of efficiency is increasing with the number of agents and the value of $\alpha$.

Bounds are very strong, near-tight.

# Example for four agents

Bounds Bounds on the Price of Fairness and the Price of Efficiency of $\alpha$-Fair Allocations for $\boldsymbol{n} = \mathbf{4}$ **agents**.




$\alpha = 1$: **Maximizes fairness** while guaranteeing max ~25% loss of system efficiency.

$\alpha = 3$: **Maximizes efficiency** while guaranteeing maximum ~20% drop in the utility for the worst-off agent (compared to egalitarian allocation).

$\alpha = 2$: **Balances efficiency and fairness**

# Allocation Procedures

# Allocation Procedures

**Protocols**: What messages do agents have to exchange and in which order?

**Strategies**: What strategies may an agent use for a given protocol? How can we give incentives to agents to behave in a certain way?

**Algorithms**: How do we solve the computational problems faced by agents when engaged in negotiation?

# Centralised vs. Distributed Allocation

**Centralised case**

- A **single entity decides** on the final allocation, possibly after having elicited the preferences of the other agents.
- Example: auctions

**Distributed case**

- **Allocations emerge** as the result of a sequence of local negotiation steps.
- Such local steps may or may not be subject to structural restrictions (say, bilateral deals).

*Which approach is appropriate under what circumstances?*

# Centralised vs. Distributed Comparison

| Centralised | Distributed |
| --- | --- |
| • The **communication** protocols required are relatively **simple**.<br>• Many **results** from **economics** and **game theory** , in particular on mechanism design, can be exploited.<br>• **Powerful algorithms** for winner determination in combinatorial auctions.<br>• Possible **trust** issues.<br>• Difficult to deal with **unbounded problems**. | • Avoids **trust** issues.<br>• Inherently **scalable**.<br>• Can take an **initial allocation** into account.<br>• More natural to model **step-wise improvements** over the status quo.<br>• Can deal with **unbounded domains**.<br>• More **complex** protocols significantly more **difficult** to analyse (convergence etc.) |

➔ Auctions

# Conclusions

Solving allocation problems requires defining 1) resources, 2) agents and their preferences, 3) system/social preferences and 4) mechanism.

There is an inherent trade-off between efficiency and fairness in allocation.

Auctions are a widely adopted centralized allocation mechanism which (typically) aims to optimize efficiency and is neutral toward fairness.

Reading: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