Computational Game Theory

Voting and Social Choice

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The problem of social choice

- The set of voters $N = \{1, \ldots, n\}$ need to make a choice from the set of alternatives (candidates) $M = \{1, \ldots, m\}$
- Each voter $i \in N$ has a different preference $\succ_i$ over M

The main question

How should a central authority pool the preferences of voters so as to best reflect the wishes of the population as a whole?

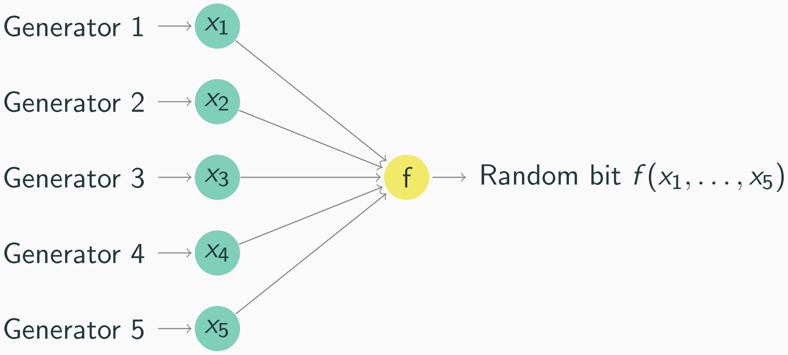
We will discuss these two situations separately:

1. $m = 2$ alternatives
2. $m \geq 3$ alternatives

An example from computer science

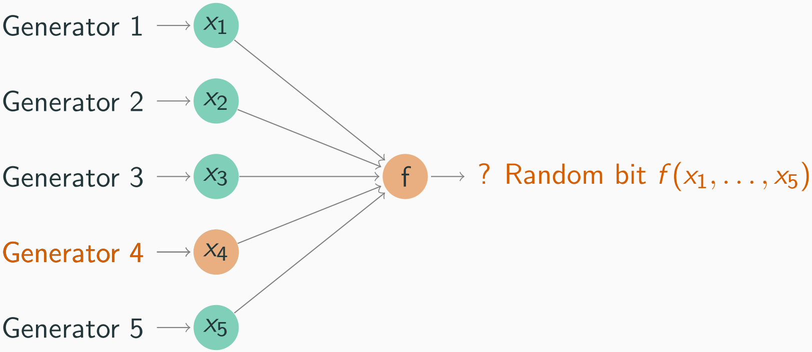
Collective Coin Flipping (M. Ben-Or and N. Linial)

Generate a random bit based on n random bits in a distributed environment without simultaneous computations



What if one of the generators is faulty?

An example from computer science (ctnd.)



Find a balanced boolean function f
that is robust in spite of corrupted inputs!

The Choice Between 2 Alternatives

Election with 2 candidates and n voters

- There are 2 candidates denoted by 0 and 1
- A vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in \{0,1\}^n$ of voters' choices

Voting rule (Social choice function)

Any boolean function

$$f\colon \{0,1\}^n \to \{0,1\}$$

mapping the votes of voters to the winner of the election

Examples of voting rules

Majority (for n odd)

$$\mathrm{Maj}_n(\boldsymbol{x}) = \begin{cases} 1 & x_1 + \cdots + x_n > \frac{n}{2} \\ 0 & \text{otherwise} \end{cases}$$

Weighted majority (weights $w_1, \ldots, w_n \geq 0$, quota $q \geq 0$)

$$f(\boldsymbol{x}) = \begin{cases} 1 & w_1 x_1 + \cdots + w_n x_n > q \\ 0 & \text{otherwise} \end{cases}$$

where $w_1 x_1 + \cdots + w_n x_n \neq q$ for all $\boldsymbol{x}$

Unanimity rule

$$\mathrm{AND}(\boldsymbol{x}) = x_1 \cdots x_n$$

Examples of voting rules (ctnd.)

At-least-one-rule

$$\mathrm{OR}(\boldsymbol{x}) = 1 - (1 - x_1) \cdots (1 - x_n)$$

Dictator rule with dictator i

$$\mathrm{Dict}_i(\boldsymbol{x}) = x_i$$

Tribe rule for tribes of size k

$$\mathrm{Tribe}(\boldsymbol{x}) = \mathrm{OR}\big(\mathrm{AND}(x_1, \ldots, x_k), \ldots, \mathrm{AND}(x_{n-k+1}, \ldots, x_n)\big)$$

Axioms for voting rules f

For all $\boldsymbol{x}, \boldsymbol{y} \in \{0,1\}^n$:

Monotonicity	$\boldsymbol{x} \leq \boldsymbol{y}$ implies $f(\boldsymbol{x}) \leq f(\boldsymbol{y})$
Neutrality	$f(\mathbf{1} - \boldsymbol{x}) = 1 - f(\boldsymbol{x})$
Symmetry	$f\left(x_{\pi(1)}, \ldots, x_{\pi(n)}\right) = f(\boldsymbol{x})$
Unanimity	$f(\mathbf{1}) = 1$ and $f(\mathbf{0}) = 0$

From axioms to voting rules

The majority function possesses all of the mathematical properties that seem desirable in a voting rule:

Theorem (May, 1952)

Let $f\colon \{0,1\}^n \to \{0,1\}$ be an arbitrary voting rule.
The following are equivalent:

- f is symmetric, monotone, and neutral.
- n is odd and $f = \mathrm{Maj}_n$.

Voting rules as simple coalitional games

There is a one-to-one correspondence between monotone unanimous voting rules f and simple coalitional games v:

$$\boldsymbol{x} \in \{0,1\}^n \longmapsto A_{\boldsymbol{x}} = \{i \in N \mid x_i = 1\}$$

$$A \subseteq N \longmapsto (\boldsymbol{x}_A)_i = \begin{cases} 1 & i \in A \\ 0 & i \notin A \end{cases}$$

From f to v

Let $v(A) = f(\boldsymbol{x}_A) \quad \forall A \subseteq N$

From v to f

Let $f(\boldsymbol{x}) = v(A_{\boldsymbol{x}}) \quad \forall \boldsymbol{x} \in \{0,1\}^n$

We can use the Banzhaf value to express voters' influence.

Does one's vote make the difference?

Impartial Culture Assumption

The preferences of voters are uniformly random indepedendent random variables $\mathbf{X} = (X_1, \ldots, X_n)$.

Specifically: $P[\mathbf{X} = \boldsymbol{x}] = \binom{n}{x_1 + \cdots + x_n} 2^{-n}$

Definition

The influence of a voter $i \in N$ on a voting rule f is the probability that the voter overturns the election outcome:

$$\mathrm{Inf}_i(f) = P\left[f(\mathbf{X}_{i=0}) \neq f(\mathbf{X}_{i=1})\right]$$

where $\mathbf{X}_{i=0} = (X_1, \ldots, X_{i-1}, 0, X_{i+1}, \ldots, X_n)$

Examples of voting rules and influence

Unanimity rule

$$I_i(\text{AND}) = \frac{1}{2^{n-1}} \qquad \forall i \in N$$

Dictator rule with dictator j

$$I_i(\text{Dict}_j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases} \qquad \forall i \in N$$

Majority

$$\text{Inf}_i(\text{Maj}_n) = \frac{\binom{n-1}{\frac{n-1}{2}}}{2^{n-1}} = \Theta\left(\frac{1}{\sqrt{n}}\right) \qquad \forall i \in N$$

Which voting rule minimizes the influence of voters?

Definition

We call a voting rule unbiased (or balanced) if $\mathbb{E}[f(\mathbf{X})] = \frac{1}{2}$.

- The unanimity rule has influence $\frac{1}{2^{n-1}}$, but it is biased as

$$\mathbb{E}[\text{AND}(\mathbf{X})] = \frac{1}{2^n}$$

- The majority rule is unbiased with influence $\Theta\left(\frac{1}{\sqrt{n}}\right)$

Is there a rule with uniformly smaller influence?

Influence of the tribe rule

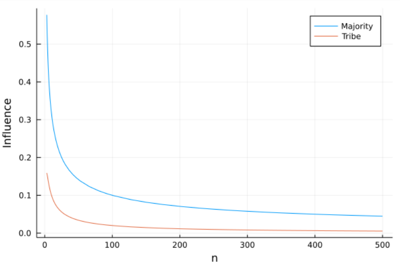
Tribe rule for some size k of tribes

$$\text{Tribe}(\boldsymbol{x}) = \text{OR}\big(\ \text{AND}(x_1, \ldots, x_k), \ldots, \text{AND}(x_{n-k+1}, \ldots, x_n)\ \big)$$

$$\max_{i \in N} \ \text{Inf}_i(\text{Tribe}) = O\left(\frac{\log n}{n}\right)$$

Kahn, Kalai, Linial (1988)

There is no unbiased voting rule with smaller influence.



The Choice Between m Alternatives

Comparing more than two alternatives

If $m = 2$, then each voter simply indicates the preferred alternative. Let $m \geq 3$.

- We denote the alternatives by $a, b, c, \ldots$
- Each voter i has a ranking $\succ_i$ over M, for example,

$$b \succ_i a \succ_i c$$

- Such a ranking should satisfy some natural axioms. . .

Preference relations

Definition

A strict preference relation over the alternatives $M = \{1, \ldots, m\}$ is a binary relation $\succ$ on M with the properties:

Completenes	$a \succ b$ or $b \succ a$	$\forall a, b \in M,\ a \neq b$
Transitivity	$a \succ b$ and $b \succ c$ implies $a \succ c$	$\forall a, b, c \in M$
Irreflexivity	$a \not\succ a$	$\forall a \in M$

Let $\mathcal{L}_M$ be the set of all strict preference relations on M.

Observe that there are no ties between alternatives, since either $a \succ b$ or $b \succ a$, but not both!

Condorcet winner

It seems natural to check if there is a candidate defeating all other candidates in a head-to-head competition:

Definition

A Condorcet winner is the candidate $c \in M$ such that $\forall a \in M$,

$$|\{i \in N \mid c \succ_i a\}| \ > \ |\{i \in N \mid a \succ_i c\}|$$

- The Condorcet winner is unique, if it exists
- We are seeking a voting rule picking the Condorcet winner

Example (Borda, 1784)

A committee composed of 21 members needs to select one individual among 3 candidates a, b, c. The preferences are:

Members	*Preference*
1	$a \succ_1 b \succ_1 c$
7	$a \succ_2 c \succ_2 b$
7	$b \succ_3 c \succ_3 a$
6	$c \succ_4 b \succ_4 a$

The Condorcet winner is c.

Example (Condorcet, 1785)

A committee composed of 60 members needs to select one individual among 3 candidates a, b, c. The preferences are:

Members	*Preference*
23	$a \succ_1 b \succ_1 c$
2	$b \succ_2 a \succ_2 c$
17	$b \succ_3 c \succ_3 a$
10	$c \succ_4 a \succ_4 b$
8	$c \succ_5 b \succ_5 a$

There is no Condorcet winner! The result of pairwise comparison is $a \succ b$ by $33 : 27$, $b \succ c$ by $42 : 18$, and $c \succ a$ by $35 : 25$.

Social choice functions

The voting rules map the voters' preference profiles to the (set of) winning candidates(s):

Definition

A social choice correspondence is a mapping $f\colon \mathcal{L}_M^n \to \mathcal{P}(M)$.
A social choice function is a mapping $f\colon \mathcal{L}_M^n \to M$.

For example, let $m = n = 3$, and suppose that

$$\begin{array}{lcl} a \succ_1 b \succ_1 c & & \\ a \succ_2 c \succ_2 b & \longmapsto & a = f(\succ_1, \succ_2, \succ_3) \\ c \succ_3 b \succ_3 a & & \end{array}$$

This seems sensible. But we need to define f for all 6^3 inputs!

Examples (Nonranking methods)

Plurality voting

Each voter casts a single vote for one candidate.

Cumulative voting

Each voter distributes k votes arbitrarily.

Approval voting

Each voter casts a single vote for possibly more candidates.

- In all the methods, the candidate with the most votes win
- The unique winner is selected by a tie-breaking rule

Examples (Ranking methods)

Plurality with elimination

Each voter casts one vote for their top candidate. The candidate with the fewest votes is eliminated. Each voter who voted for the eliminated candidate casts a new vote for one of the remaining candidates. This is repeated until only one candidate remains.

Borda voting

If an alternative is ranked as the i-th highest by a voter, it receives $m - i$ points from that voter. The winning alternatives maximize the total sum of points from all the voters.

Desirable properties of social choice functions

Let $f : \mathcal{L}_M \to M$ be a social choice function. We say that f is

- unanimous if for any $a \in M$ and every $\succ \in \mathcal{L}_M^n$:

$$\forall i \in N \, \forall b \in M \setminus a : \quad a \succ_i b \quad \Rightarrow \quad f(\succ) = a$$

- monotone if, for all $\succ, \succ' \in \mathcal{L}_M^n$ satisfying the condition $a \succ_i b \Rightarrow a \succ'_i b$ for all different $a, b \in M$ and each $i \in N$:

$$f(\succ) = a \quad \Rightarrow \quad f(\succ') = a.$$

Which social choice functions have desirable properties?

Theorem (Muller–Satterthwaite, 1977)

Let $m \geq 3$ and f be any unanimous and monotone social choice function. Then f is dictatorial, that is, there exists a voter $i \in N$ such that f always selects the top choice of i.

- The theorem implies that *all* non-dictatorial social choice functions lack unanimity or monotonicity
- This is an instance of the famous Arrow's theorem (1951)