Computational Game Theory

Coalitional Games and the Core

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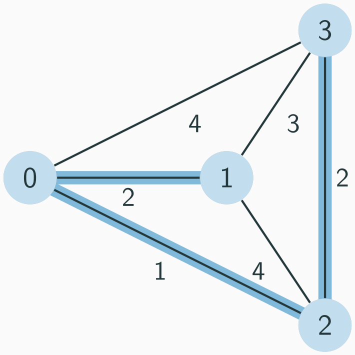
How to share the cost?

- *Cities* $1, 2, 3$ need to connect to the *provider* of energy 0
- The graph shows costs of pairwise connections
- For each $A \subseteq \{1, 2, 3\}$, the cost of connecting A to 0 is determined by a minimum cost spanning tree on $A \cup \{0\}$



$$c(A) = \begin{cases} 1 & A = 2 \\ 2 & A = 1 \\ 4 & A = 3 \\ 3 & A = 12, 23 \\ 5 & A = 13, 123 \end{cases}$$

How to determine the voting power?

UN Security Council

- 5 *permanent* and 10 *non-permanent* members
- A binary decision is approved by all the permanent members and ≥ 4 non-permanent members
- There are 2^{15} voting scenarios!

What is it all about?

- Fair division of costs
- Power of agents controlling some resources
- Fairness of a complicated voting system
- Efficient allocation of the profit among agents

Game forms

1. Normal (Strategic)
2. Extensive
3. Coalitional

Games in coalitional form

- Players can form *coalitions*
- A coalition is a set of players coordinating their strategies in order to maximize *the utility of the coalition*
- Strategic aspects of coalitional games are unimportant, since they are implicitly part of the deal among players

Players and coalitions

- The player set is

$$N = \{1, \ldots, n\}, \quad \text{for some } n \in \mathbb{N}$$

- A coalition is a subset $A \subseteq N$, where
  - $\emptyset$ is the empty coalition
  - N is the *grand coalition*
  - $\{i\}$ is a one-player coalition
- The set of all coalitions is the powerset

$$\mathcal{P}(N) = \{A \mid A \subseteq N\}$$

Coalitional games

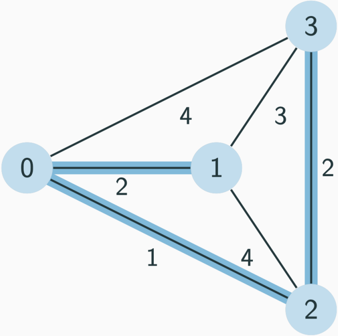
Definition

Coalitional game is a pair (N, v), where v is a function

$$v\colon \mathcal{P}(N) \to \mathbb{R} \qquad \text{such that } v(\emptyset) = 0.$$

- The players in coalition A receive the worth $v(A)$ independently of the actions of players in $N \setminus A$
- We will identify a coalitional game (N, v) with function v when the player set N is understood

Example: Savings game v



$$c(A) = \text{cost of connecting } A$$

$$v(A) = \sum_{i \in A} c(i) - c(A)$$

$$= \begin{cases} 0 & A = \emptyset, 1, 2, 3, 12 \\ 1 & A = 13 \\ 2 & A = 23, 123 \end{cases}$$

Game v is superadditive:

$$v(A) + v(B) \leq v(A \cup B) \qquad \text{if } A \cap B = \emptyset.$$

Example: Voting game

UN Security Council

- 5 *permanent* and 10 *non-permanent* members
- A binary decision is approved by all the permanent members and ≥ 4 non-permanent members

$$v(A) = \begin{cases} 1 & \text{if } A \supseteq \{1, \ldots, 5\} \text{ and } |A| \geq 9, \\ 0 & \text{otherwise.} \end{cases}$$

Game v is superadditive and simple:

- $v(A) \in \{0, 1\}$
- v is monotone and $v(N) = 1$

Main questions of coalitional game theory

1. **Which** coalitions will form?
2. **How** a coalition allocates its worth to its members?

Which coalitions will form?

- A coalitional structure is a partition $\mathcal{S} = \{A_1, \ldots, A_k\}$ of N:
  1. $A_1 \cup \cdots \cup A_k = N$, where $A_i \neq \emptyset$
  2. $A_i \cap A_j = \emptyset$ for all $i \neq j$
- The total utility of $\mathcal{S}$ is then

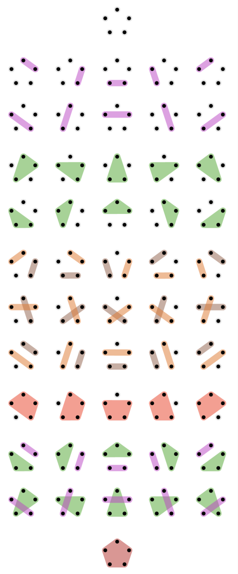
$$V(\mathcal{S}) = \sum_{i=1}^{k} v(A_i)$$

Coalition formation problem

Find a coalitional structure $\mathcal{S}^*$ satisfying

$$V(\mathcal{S}^*) = \max\{V(\mathcal{S}) \mid \mathcal{S} \text{ is a coalitional structure}\}$$

Example: Coalitional structures for five players



Source: Wikipedia

Coalition formation problem

- Bell numbers B_n count the number of coalitional structures:

n	3	...	10	...	15
B_n	15	...	115 975	...	1 382 958 545

- Finding an optimal coalitional structure $\mathcal{S}^*$ is NP-complete

Trivial solution for superadditive games

Let v be a superadditive game. For any coalitional structure $\mathcal{S}$,

$$V(\mathcal{S}) = \sum_{i=1}^{k} v(A_i) \leq v(N) = V(\{N\}).$$

This implies that $\mathcal{S}^* = \{N\}$.

Main questions revisited

Which coalitions will form?

- We assume that players form grand coalition N
- This is optimal for superadditive games

How a coalition allocates its worth to its members?

- An allocation is a vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$
- If $\boldsymbol{x}$ is allocated to players, coalition $A \subseteq N$ obtains

$$\boldsymbol{x}(A) = \sum_{i \in A} x_i$$

The solution of a game v is some set of allocations $\boldsymbol{x} \in \mathbb{R}^n$.

Agenda

We will study three solution concepts in this course:

1. Core
2. Shapley value
3. Nucleolus

Core

What is the core of a game?

The core is a set of efficient allocations upon which no coalition can improve.

Definition

The core of a game v is the set

$$\mathcal{C}(v) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \underbrace{\boldsymbol{x}(N) = v(N)}_{\text{Efficiency}}, \quad \underbrace{\boldsymbol{x}(A) \geq v(A), \ \forall A \subseteq N}_{\text{Coalitional rationality}}\}.$$

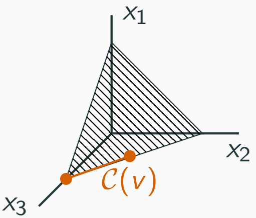
- The core is a convex polytope in $\mathbb{R}^n$ of dimension $\leq n-1$
- Is the core always nonempty? How to find core allocations?

Example: Savings game v

What is the distribution of total saving?

$$v(A) = \begin{cases} 0 & A = \emptyset, 1, 2, 3, 12 \\ 1 & A = 13 \\ 2 & A = 23, 123 \end{cases}$$

$$\begin{aligned} \mathcal{C}(v) &= \left\{ \boldsymbol{x} \in \mathbb{R}^3_+ \mid \boldsymbol{x}(12) \geq 0,\ \boldsymbol{x}(13) \geq 1,\ \boldsymbol{x}(23) \geq 2,\ \boldsymbol{x}(123) = 2 \right\} \\ &= \operatorname{conv}\left\{(0,0,2),(0,1,1)\right\} \end{aligned}$$

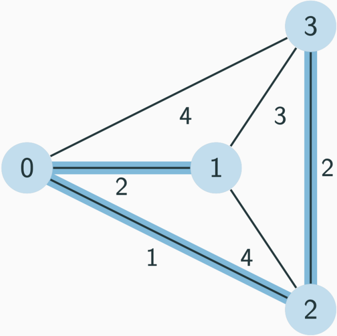


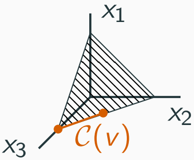
How to divide the cost?

$$\Downarrow$$

$$y_i = c(i) - x_i$$

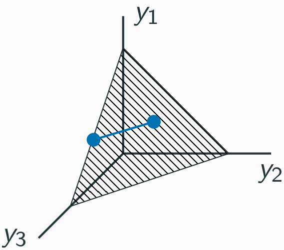
How to divide the cost – a solution





$$y_i = c(i) - x_i$$

$$\boldsymbol{y} \in \text{conv}\{(2,0,3),(2,1,2)\}$$



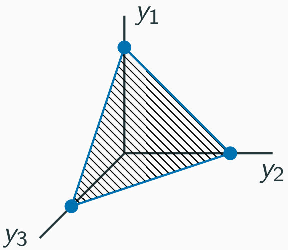
Games can have empty cores

Simple majority voting

Three players vote by majority. This determines a game

$$v(A) = \begin{cases} 1 & |A| \geq 2, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for all } A \subseteq \{1, 2, 3\}.$$

Then $\mathcal{C}(v) = \emptyset$.



How to decide nonemptiness of the core

$$\mathcal{C}(v) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x}(N) = v(N), \quad \boldsymbol{x}(A) \geq v(A),\ \forall A \subseteq N\}$$

Linear program with real variables $x_1, \dots, x_n$

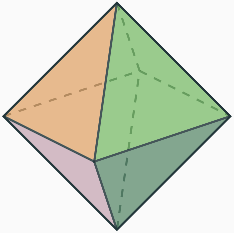
Minimize $x_1 + \cdots + x_n$

subject to $\sum_{i \in A} x_i \geq v(A)$ for each nonempty $A \subseteq N$

The following are equivalent:

- The optimal value is $v(N)$
- $\mathcal{C}(v) \neq \emptyset$

How to find all core allocations



The core $\mathcal{C}(v)$ has a representation

$$\mathcal{C}(v) = \text{conv}\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\},$$

where $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ are the vertices of $\mathcal{C}(v)$.

Vertex enumeration problem

- Find *all* vertices of the core $\mathcal{C}(v)$
- A hard problem studied in polyhedral geometry

This problem has a closed-form solution for some games.

Games with incentives to join large coalitions

A game v is supermodular if

$$v(A) + v(B) \leq v(A \cup B) + v(A \cap B) \qquad \text{for all } A, B \subseteq N.$$

Proposition

The following are equivalent.

- Game v is supermodular.
- For all $A, B \subseteq N$ with $A \subseteq B$, and each $i \in N \setminus B$,

$$v(A \cup i) - v(A) \leq v(B \cup i) - v(B).$$

It is about marginal contributions of players

- Given a permutation π of N, the rank of player i is $\pi(i)$
- The coalition preceding player i is then

$$A_i^\pi = \{j \in N \mid \pi(j) < \pi(i)\}.$$

Definition

A marginal vector is an allocation $\boldsymbol{x}^\pi \in \mathbb{R}^n$ such that

$$x_i^\pi = v(A_i^\pi \cup i) - v(A_i^\pi), \qquad i \in N.$$

Example: Marginal vectors in a supermodular game

This three-player game is supermodular: $v(A) = \begin{cases} 0 & |A| \leq 1, \\ 1 & |A| = 2, \\ 3 & |A| = 3. \end{cases}$

Permutation	**Marginal vector**
123	$(0, 1, 2)$
132	$(0, 2, 1)$
213	$(1, 0, 2)$
231	$(2, 0, 1)$
312	$(1, 2, 0)$
321	$(2, 1, 0)$

Observe that each marginal vector is a core allocation.

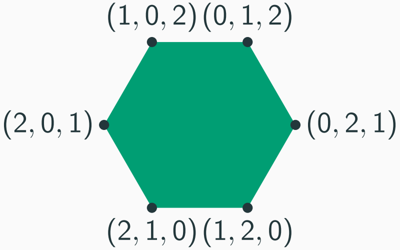
Cores of supermodular games

Theorem

The following are equivalent.

- Game v is supermodular
- The vertices of $\mathcal{C}(v)$ are precisely marginal vectors

$$v(A) = \begin{cases} 0 & |A| = 1, \\ 1 & |A| = 2, \\ 3 & |A| = 3. \end{cases}$$



Summing up the core properties

Pros

- Simple definition
- Core allocations are stable
- Known for some games

Cons

- May be empty
- May be large
- Hard to compute

We can seek solution concepts based on different criteria:

- Nonemptiness
- Single allocation
- Fairness