Symmetric Single-Item IPV Auctions

Tomáš Votroubek Dominik Andreas Seitz

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We will think of auctions as mechanisms that collect each agent's bid, then decide what to allocate to each bidder and what they should pay for it. The goal of auctions is not necessarily to maximize revenue (those are called optimal auctions); We might instead be interested in maximizing social welfare, in which case the payment rule better be set to incentivize the bidders to reveal their valuations.

Most of the following exercises involve only auctions where the players' independent private values are drawn from the same uniform distribution on $[0, 1]$. The distribution is uniform to avoid unnecessary calculus and keep the exercises simple; the restriction to symmetry is there because it is not obvious how to even determine whether an asymmetric auction has an equilibrium.

Each problem is followed by a page containing the solution. Problems marked with a star either do not have solutions or their solutions fall outside this course's scope. Think about how you would approach such problems and what makes them difficult.

Basic definitions and terminology Suppose there are two bidders a_1 and a_2 with private values drawn i.i.d. from a uniform distribution on $[0, 1]$. For each of the examples, find the equilibrium, then answer the following:

- b_i: What are the bids of each player $i \in \{a_1, a_2\}$.
- $x_i(\mathbf{b}), p_i(\mathbf{b})$: What is the allocation to each bidder and what do they pay for it?
- $u_i(\mathbf{b})$: What is the utility of each player.
- R: What revenue does the auction generate.

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1. The private values of the bidders are $v_1 = 0.3$ and $v_2 = 0.9$. Suppose the auction is run as a second-price sealed-bid auction.
2. The private values remain $v_1 = 0.3$ and $v_2 = 0.9$, but the auction is now run as a *first-price* auction.
3. The private values are $v_1 = 0.6$ and $v_2 = 0.9$, and the auction is run as a second-price auction.
4. As before, the private values are $v_1 = 0.6$ and $v_2 = 0.9$, but the auction is now run as a first-price auction.

- Second price auctions are *DSIC* — bidding your private value is a (weakly) dominant strategy. In a first-price auction, bidders have to take into account their beliefs of other players values: for our setting of two bidders with private values drawn i.i.d. from a uniform distribution on $[0,1]$ the Bayesian Nash equilibrium strategy is to bid a half of your value (we will see why later).

- Allocations denote which bidder wins which items, or even how much of each. In single item auctions, allocations are just 0-1 vectors with at most one 1. In both the first-price and the second-price auctions, the highest bidder wins the item. In a first-price auction, the highest bidder pays his own bid, in a second-price auction the highest bidder pays the second highest bid.

- While more complex models exist, we will consider only the quasilinear utility $u_i(\mathbf{b}) = v_i x_i(\mathbf{b}) - p_i(\mathbf{b})$. Your utility is "what you get minus what you pay for it."

- The reveune is the sum of the payments.

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1. $\mathbf{b} = (0.3, 0.9)$; $\mathbf{x} = (0, 1)$; $\mathbf{p} = (0, 0.3)$; $u_1(\mathbf{b}) = 0$, $u_2(\mathbf{b}) = 0.6$; and $R = 0.3$.

2. $\mathbf{b} = (0.15, 0.45)$; $\mathbf{x} = (0, 1)$; $\mathbf{p} = (0, 0.45)$; $u_1(\mathbf{b}) = 0$, $u_2(\mathbf{b}) = 0.45$; and $R = 0.45$.

3. $\mathbf{b} = (0.6, 0.9)$; $\mathbf{x} = (0, 1)$; $\mathbf{p} = (0, 0.6)$; $u_1(\mathbf{b}) = 0$, $u_2(\mathbf{b}) = 0.3$; and $R = 0.6$.

4. $\mathbf{b} = (0.3, 0.45)$; $\mathbf{x} = (0, 1)$; $\mathbf{p} = (0, 0.45)$; $u_1(\mathbf{b}) = 0$, $u_2(\mathbf{b}) = 0.45$; and $R = 0.45$.

Properties and relations of auctions Answer the following:

1. How would you find the strategies, if the auctions in the previous exercise were run using the rules of English, Japanese, or Dutch auctions instead?
2. Compare the revenues of the auctions in the previous exercise. Should you prefer a first-price or a second-price auction if you were a seller? What if you were a risk-averse seller?
3. Are second-price auctions always efficient? What about first-price auctions?

1. The English and Japanese auctions are *strategically equivalent* to the second-price auction, while the first-price auction is to the Dutch. The auctions might progress differently, but the equilibrium strategies ("the information you would need to give to someone else to bid on your behalf") do not change.

2. While in every realization of the bidders values the revenue was different, *on average* the revenue is the same. In fact, with a few assumptions, all efficient auctions yield the same revenue (see the *revenue equivalence theorem*). The answer changes for risk-averse sellers, however. The highest bidder's payment in a second-price auction depends only on the second-highest bid (which can be anything between 0 and 1). In contrast, the payments in our first-price auction range from 0 to 0.5. While the second-price auction might yield a higher revenue in some cases, a risk-averse seller would like to avoid low payoffs and would prefer the "more stable" first-price auction.

3. Pareto efficiency in single-item auctions[1] reduces to: "does the item end up with the person who wants it the most?"[2] Second-price auctions are actually *designed* to be welfare-maximizing, more on that next week. In first-price symmetric auctions, all the bidders will "speculate" about the values of others "symmetrically", so the highest bid must always be made by the bidder with the highest valuation. Asymmetric first-price auctions, on the other hand, are not always efficient.

[1] "Not leaving money on the table for unaccounted side-payments".
[2] Also "social-wellfare maximization." Contrast with the prisoner's dilemma.

Discrete private values Suppose there are two bidders a_1 and a_2, whose independent private values are either 1 or 3 with equal probability. Assume ties are broken randomly.

1. Find the expected revenue of the auction above.
2. ★ What would be the equilibrium of a first-price auction?
3. Now suppose there were three bidders instead of two. How does the revenue change?

The simplest thing to do here is to enumerate all of the realizations:

	v_1	v_2	R
25 %	1	1	1
25 %	1	3	1
25 %	3	1	1
25 %	3	3	3

which add up to an expected revenue of 1.5. The same method can be used to show that the three-player auction yields a revenue of 2.

In the case of the first-price auction, the task is not so simple. There is no formula to calculate the equilibrium strategies like there was for the continuous uniform distribution. In fact, due to the discreteness, equilibria will be in mixed strategies.[3]

[3] Add asymmetry and there may be no equilibrium at all!

Auctions as EFG trees : Consider a second-price auction involving two bidders a_1 and a_2 whose private values are either 0 or 1 with equal probability. Bidder a_1 sometimes makes a mistake about his value for the object: when his value is 1, he knows it is 1; however, when his value is 0, half of the time, he believes it is actually 1 by mistake. Assume that ties are broken randomly and that bids must be integers. Draw the full game tree of this situation.

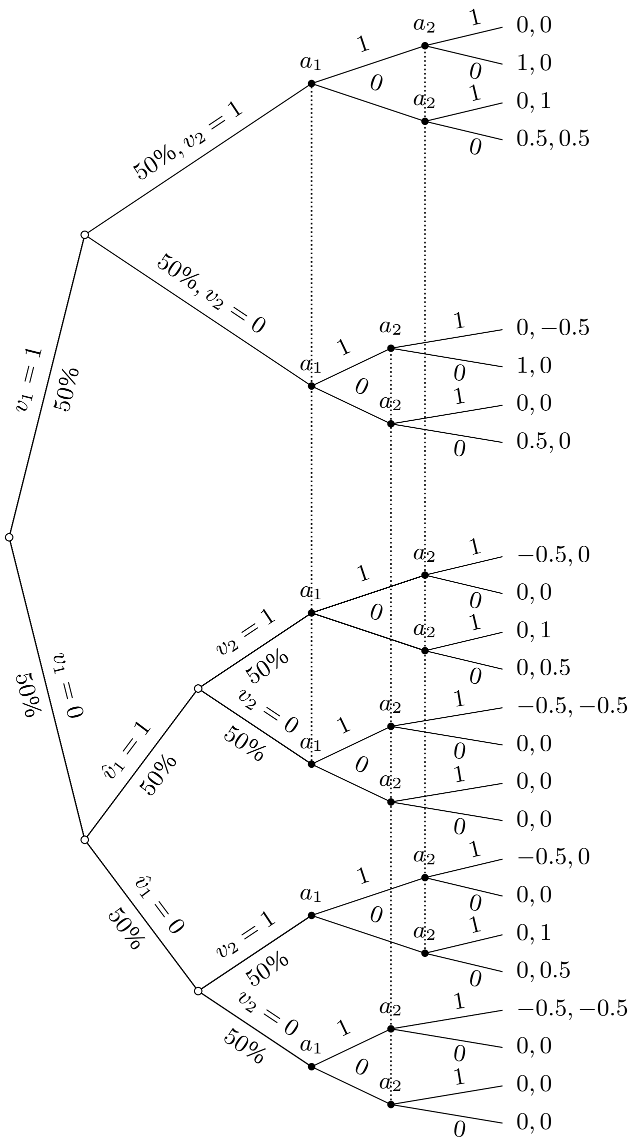

Figure 1: The game tree with expected payoffs instead of an explicit tie-breaking round. There are multiple correct solutions.

Auctions as EFGs cont. Using the tree from the last exercise, calculate a_1's expected utility from bidding 0 compared to bidding 1, when a_1 thinks his value is 1. What is a_1's optimal strategy?

Start by assuming that a_2 plays rationally and bids truthfully (if you drew branches for other possibilities at all), then for each of a_1's bidding strategies, find the possible outcomes and what their probabilities are.

The expected payoff of bidding 0 is 0.125, compared to a payoff of 0.1875 for bidding 1, when a_1 thinks his value is 1. Bidding truthfully (with respect to observed values) is still the optimal strategy.

Equilibria in first-price auctions ★ Consider a first-price auction with n bidders with private values drawn i.i.d. from a uniform distribution on $[0, 1]$. What is the equilibrium strategy?[4]

[4] Hint: use the revenue equivalence theorem.

There are multiple ways of approaching this. We can use the revenue equivalence theorem and reason, that if the expected revenue of the second-price auction and the first-price auction are to be the same on average, then the strategy of the winning bidder in a first-price auction better result in an expected payment equal to the second highest valuation — the revenue of the second-price auction.

Since we are interested in only the strategy of the winner (no one else contributes to the revenue), we are looking for the next highest bid below the winner's. That is, we are looking for the highest valuation of $n-1$ between 0 and the winner's value. This is known as an order statistic[5] and has a nice formula for uniform distributions:

$$\text{The } k\text{th largest of } m \text{ i.i.d. draws from } [0, x] \text{ is } \frac{m+1-k}{m+1}x.$$

Plugging our numbers into the formula, we get that the equilibrium strategy of the winner (and indeed all the players, since no one knows who is going to win beforehand) with a value v is $\frac{(n-1)+1-1}{(n-1)+1}v$ or $\frac{n-1}{n}v$.

You can fix the strategies of all other players in the profile and verify that this really is an equilibrium.

[5] At least the MAS book calls it that. Wikipedia, on the other hand, says "The kth order statistic of a statistical sample is equal to its kth-**smallest** value."

Cheating and collusion Consider a second-price sealed-bid auction with n bidders:

1. Explain the problem with sending sealed-bids to a seller interested in maximizing their revenue?
2. How might the bidders benefit from cooperating in a second-price auction?

1. If the auction is run by a party interested in maximizing revenue, they are incentivezed to submit a false bid close to the winning bid. That way the winner of the second-price is essentially participating in a first-price auction unknowingly (and with a terrible strategy).

2. The second-price auction has a cheating problem in general (inherited from its VCG roots). The book mentions at least the concept of a bidding-ring, which prevents all but the highest bid of the participants from reaching the seller, thereby decreasing his revenue, and then redistributes this "lost" revenue amongst its participants.[6]

[6] Would this work with only two bidders? Should you involve every bidder in your ring? Would the result still be truthful (think about the payments)?

Reserve prices Consider a second-price auction with two bidders with independent private values sampled from a uniform distribution over $[0, 1]$.

1. Calculate the expected revenue with no reserve price.
2. How would the revenue change, if the reserve price was 0.5?
3. How high should you set the reserve price to maximize your revenue?

1. The payment and the revenue in a second-price auction equals the second highest bid. Using order statistics to find the second highest draw of two draws from a uniform distribution on $[0, 1]$ results in $\frac{n+1-k}{n+1}x = \frac{2+1-2}{2+1}1$, or $R = \frac{1}{3}$.

2. There are four possibilities, all with equal probability:
   - either both bids are below the reserve, yielding a revenue of 0;
   - **2x**: only the first or only the second players value is above the reserve, in both cases the revenue is the reserve price of a half;
   - or both players bids are above the reserve, in which case the second highest of them (and expected revenue as a result) is $\frac{2}{3}$.

   All in all, the expected revenue is $R = \frac{5}{12}$.

3. The reserve price of $\frac{1}{2}$ is optimal. Either redo the calculation leaving the reserve as a free variable, then differentiate to find the maximum, or use Meyerson's theorem and solve $r - \frac{1-F(r)}{f(r)} = 0$ (see *virtual valuation*).