Combinatorial Auctions

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Today:

- Combinatorial auctions
- Vickrey-Clarke-Groves (VCG) mechanism
- Bidding Languages

Your value function is:

$$v_{you} = \begin{cases} 10, & \text{if you win A and B} \\ 0, & \text{otherwise} \end{cases}$$

There are two *simultaneous* English auctions. How should you bid?[1]

	auction for A	auction for B
bid_{me}	1	
bid_{you}	?	?

[1] You do not know my value function.

If my value function was $v_m(A) = v_m(B) = v_m(AB) = 7$

("I want to win one")

then you lose value by participating.

Given that Your value function was

$$v_y(AB) = \boxed{10}, \quad v_y(A) = v_y(B) = 0;$$

while mine was

$$v_m(A) = v_m(B) = v_m(AB) = \boxed{7};$$

what *ought to* happen?[2]

[2] Which allocation would be efficient?

Combinatorial Auctions More Formally

Combinatorial auction are mechanisms for allocating multiple goods, parameterized by:

- A set of *bidders* $N = \{a_1, \ldots, a_n\}$;
- A set of *goods* $G = \{g_1, \ldots, g_m\}$;
- and *valuation functions* $v_i : \mathcal{P}(G) \to \mathbb{R}, \quad \forall i \in N$, s.t. $v_i(\emptyset) = 0$.

Their outcomes are defined by *payments* $\mathbf{p}$ and *allocations* $\mathbf{x}$

$$\Omega = \left\{ (\mathbf{x}, \mathbf{p}) \middle| \begin{array}{l} \mathbf{x} = (x_1, \ldots, x_n), \ \bigsqcup_{i \in N} x_i \subseteq G; \\ \mathbf{p} \in \mathbb{R}^n \end{array} \right\}.$$

The *utility* of each bidder is[3] $u_i(\mathbf{x}, \mathbf{p}) = v_i(x_i) - p_i$.

[3] No externalities in auctions!

Value functions

Given two subsets $G_1, G_2 \subseteq G$, s.t. $G_1 \cap G_2 = \emptyset$ (or allocations), valuation functions can exhibit two[4] behaviors:

Substitutability When $v(G_1 \cup G_2) < v(G_1) + v(G_2)$.
As in "I only really need one."

Complementarity When $v(G_1 \cup G_2) > v(G_1) + v(G_2)$.
As in "What am I going to do with just one shoe?"

Such functions are called *subadditive* and *superadditive*, respectively.

[4] Excluding the uninteresting case of additivity.

Back to our simultaneous auctions

	$auction_A$	$auction_B$
b_m	1	
b_y	?	?

Is it really that bad?

You could do a lot worse[5]

[5] At least you could see the auction evolve.

Table 5
First- and second-bid disparities.

Type of license	First & second bids
UHF TV channel, Christchurch area	$ 100,004
	$ 6
UHF TV channel, North Island	$ 107,000
	$ 2,000
FM Radio channel, South Island	$ 35,070
	$ 159
FM Radio channel, Wellington area	$ 550,111
	$ 159
TACS B cellular management right	$7,000,000
	$ 5,000

Technically, they were *not* trying to maximize revenue…
We will not either, but let us at least see how to do it properly.

Brief detour from auctions

Should we build a road?[6]

	build	not build	payment under VCG
a_1	200	0	
a_2	100	0	
a_3	0	250	

[6] Example from Game Theory Online.

Brief detour from auctions

Should we build a road?[6]

	build	not build	payment under VCG
a_1	200	0	150
a_2	100	0	50
a_3	0	250	0

[6] Example from Game Theory Online.

The connection with auctions?

Under VCG bidders pay their "*social cost*,"
just like in the second-price (Vickrey) auction!

Collusion

What if both increase their bids?

	build	not build	payment
a_1	250	0	
a_2	150	0	
a_3	0	250	

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	build	not build	payment
a_1	250	0	100
a_2	150	0	0
a_3	0	250	0

Monotonicity and pretending to be two bidders

Compare

	build	not build	payment
a_1	20	0	10
a_2	0	10	0

with[7]

	build	not build	payment
a_1	20	0	
a_1	20	0	
a_2	0	10	0

[7] Luckily, in an auction, you can not bid for someone *else* to win.

Monotonicity and pretending to be two bidders

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	build	not build	payment
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with[7]

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a_1	20	0	0
a_1	20	0	0
a_2	0	10	0

[7] Luckily, in an auction, you can not bid for someone *else* to win.

Back to auctions.

VCG auction example 1

Find the social wellfare maximizing allocation and the corresponding payments under VCG.

	v_i			
	A	B	AB	payment
$bidder_1$	10	5	15	
$bidder_2$	1	6	12	

VCG auction example 1

Find the social wellfare maximizing allocation and the corresponding payments under VCG.

	v_i			
	A	B	AB	payment
$bidder_1$	10	5	15	6
$bidder_2$	1	6	12	5

VCG auction example 2

Find the social wellfare maximizing allocation and the corresponding payments under VCG.

	v_i			
	A	B	AB	payment
$bidder_1$	10	5	15	
$bidder_2$	1	10	12	

VCG auction example 2

Find the social wellfare maximizing allocation and the corresponding payments under VCG.

	v_i			
	A	B	AB	payment
$bidder_1$	10	5	15	2
$bidder_2$	1	10	12	5

Bidding languages.

Asking bidders for exponentially many bids is impractical.

Atomic bids

Let the bidders decide on the bundles. Represents an *AND* operator.

$$(\text{“Left glove”} \wedge \text{“Right glove”}, 20)$$

“I want a left glove *AND* a right glove,
but have no use for them individually.”

OR bids

A disjunction of atomic bids. Can not directly represent substitutability! Represents the logical OR in the sense of "at least one", not the english intuitive interpretation.

$$(F, 30) \vee (D, 20)$$

"I would pay 30 for food *OR* 20 for drink,
both are worth 50 to me."[8]

$$(v_1 \vee v_2)(S) = \max_{R \sqcup T \subseteq S} v_1(R) + v_2(T)$$

[8] I know, I know...*You* try coming up with an example, then!

XOR bids

Represents an exclusive OR of atomic bids, i.e. “at most one.”

$$(R, 300) \oplus (C, 200)$$

“I would pay 300 for a train ticket R *XOR* 200 for ticket C,
I can not ride two trains at once.”

$$(v_1 \oplus v_2)(S) = \max(v_1(S), v_2(S))$$

OR^* bids

You can simulate XOR with OR, by including *dummy* variables.

$$(R, 300) \oplus (C, 200)$$

is equivalent to

$$(R \wedge \mathcal{D}, 300) \vee (C \wedge \mathcal{D}, 200)$$

as you can not satisfy both atomic bids at the same time.

Bidding Language Example: OR

Consider the following OR bid:

$$(A \wedge B, 7) \vee (D \wedge E, 8) \vee (A \wedge C, 4)$$

What valuations does the bid express?[9]

Allocation	Value
A	
AB	
AC	
ABC	
ABDE	

[9] Assume *free disposal* and *nothing-for-nothing*.

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Allocation	Value
A	0
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What valuations does the bid express?

Allocation	Value
A	0
AB	7
AC	4
ABC	7
ABDE	8

Have a nice day!