Robust Adaptive Floating-Point Geometric Predicates

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+ additional notes by Petr Felkel, CTU Prague, 2020

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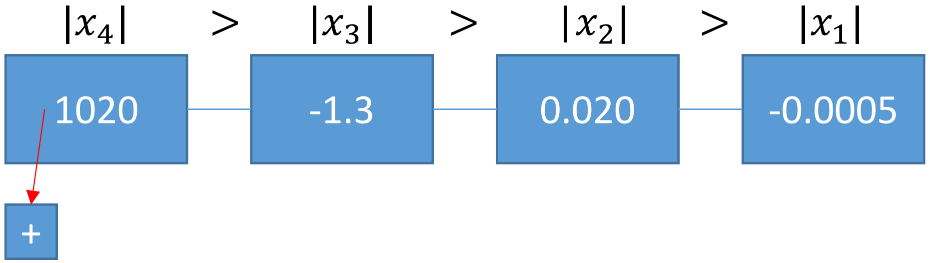
Expansion

- **Sorted** sequence of **non-overlapping** machine native numbers (float, double) – each with its own exponent and significand (mantissa)
- Sorted by **absolute values**
- **Signum** of the highest FP number is the signum of the expansion
- Zero members of the expansion will be not added.

```
1020
-   1.3
-------
1018.7
+   0.02
---------
1018.7200
-   0.0005
---------
1018.7195
```



$|x_4| > |x_3| > |x_2| > |x_1|$

1020 -1.3 0.020 -0.0005

+

represents $x = +1018.7195$

approximated $x \sim +1020 = x_4$

Expansions are not unique

binary	decimal
$1100 \ + \ (-10.1)$	$\ldots 12 + (-2.5)$
$= \ 1100.0 - 10.1$	$\ldots 12 \ - 2.5$
$= \ 1001 \ + \ 0.1$	$\ldots \quad 9 + 0.5$
$= \ 1000 \ + \ 1 \ + \ 0.1$	$\ldots \quad 8 + 1 + 0.5$
All represent the value 1001.1	$\ldots 9.5$

Meaning of symbols

p-bit floating point operations with exact rounding (float, double):

$\oplus$ addition

$\ominus$ subtraction

$\otimes$ multiplication

Exact rounding

Operations with exact rounding to p-bits (32 / 64) store result:

exact results store exact, and
non-precise results store rounded

More than 4-bits arithmetic

$$010 \times 011 \;=\; 100$$
$$2 \times 3 = 6$$

$$111 \times 101 \;=\; 100011$$
$$7 \times 5 = 35$$

With exact rounding to 4-bits

$$010 \otimes 011 \;=\; 100$$
$$2 \otimes 3 = 6$$

$$111 \otimes 101 \;=\; 1.001 \times 2^5$$
$$7 \otimes 5 \;= 36$$

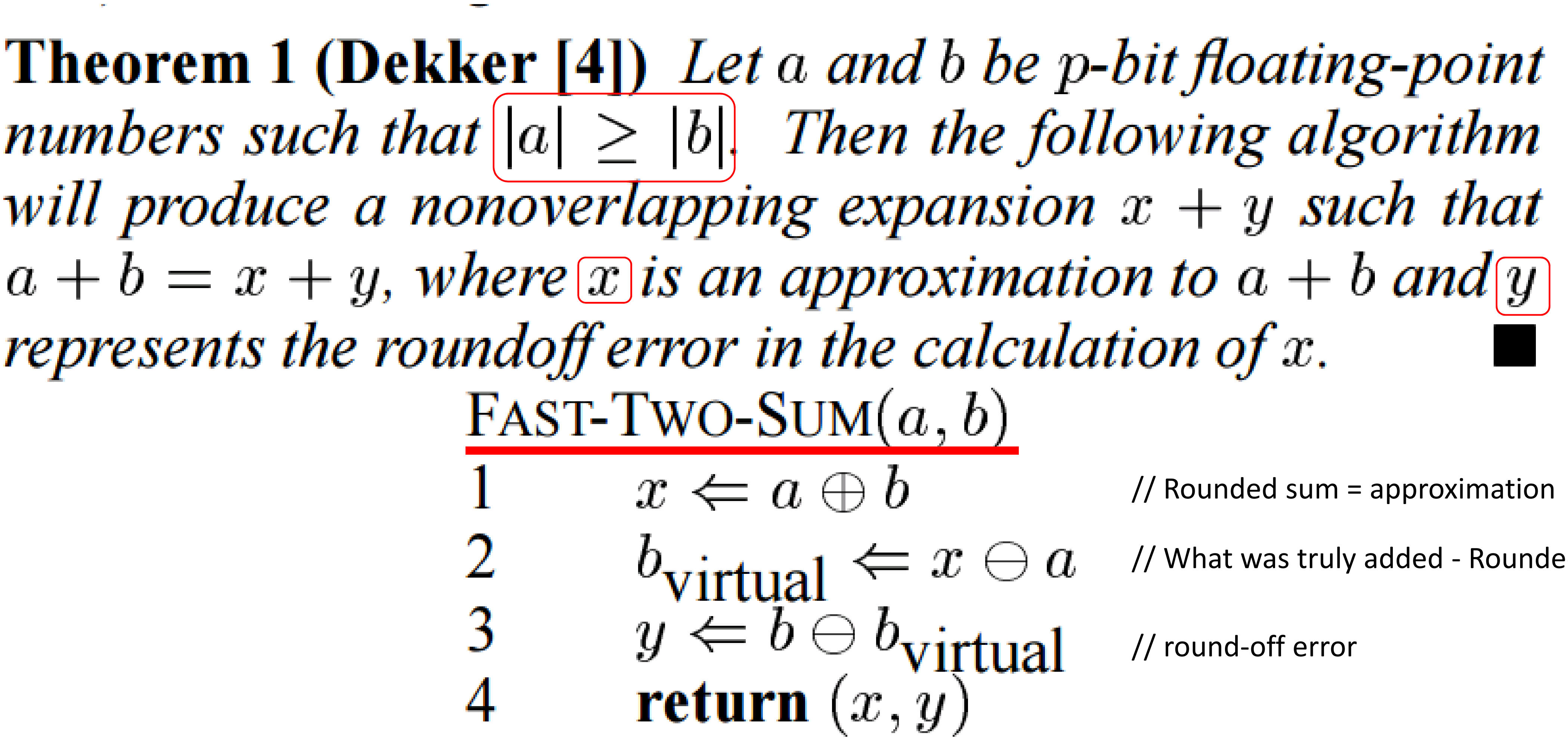
```
if (possible)
    store exact
else
    store rounded
```

Operations on expansions

IEEE 754 standard on floating point format and computing rules.

Operations on expansions require *exact rounding* of each op. to 32 / 64bit.

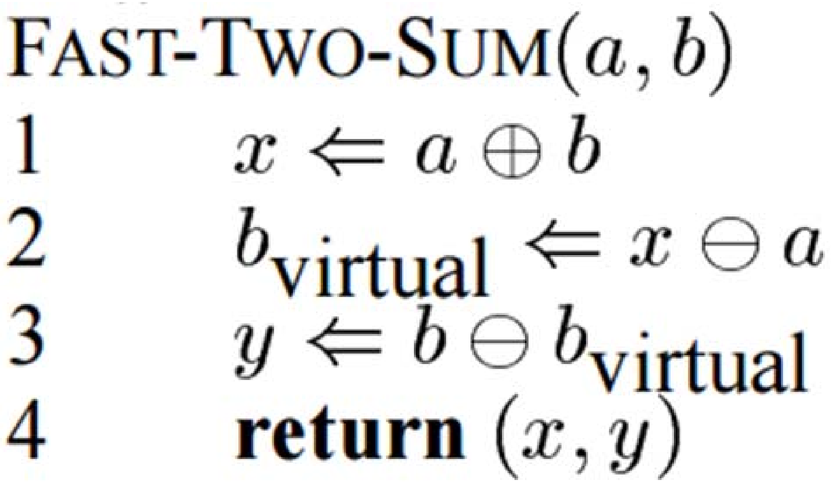
```
Fast-Two-Sum:  (a>=b) -> (x, y),         a+b=x+y
Two-Sum (a, b) -> (x, y)
Linear-Expansion-Sum (exp_a interleaved with exp_b) -> expansion

Split (a) -> (a_hi, a_lo),           a=a_hi+a_lo
Two-Product (a,b) -> (x, y)
```



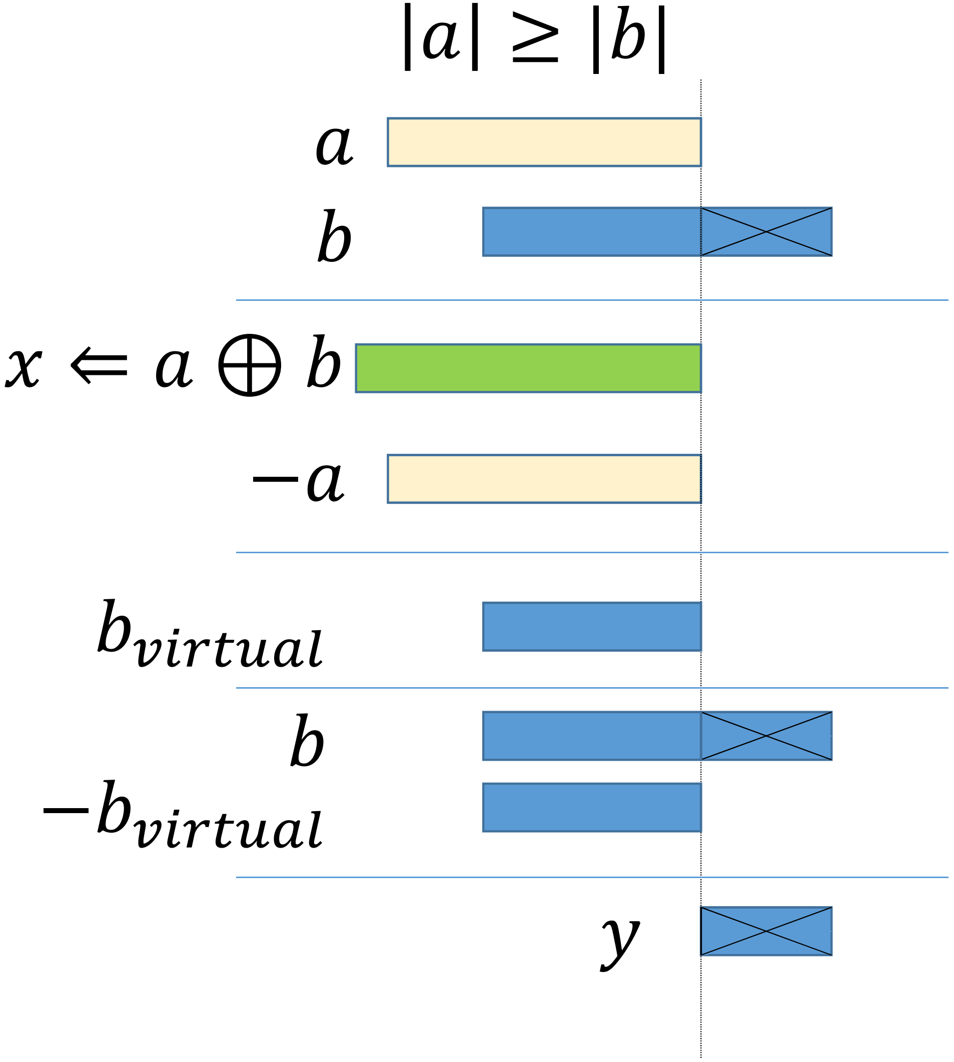
Theorem 1 (Dekker [4]) *Let a and b be p-bit floating-point numbers such that $\boxed{|a| \geq |b|}$. Then the following algorithm will produce a nonoverlapping expansion $x + y$ such that $a + b = x + y$, where $\boxed{x}$ is an approximation to $a + b$ and $\boxed{y}$ represents the roundoff error in the calculation of x.* ■

FAST-TWO-SUM(a, b)

```
$x \Leftarrow a \oplus b$                          // Rounded sum = approximation
$b_{\text{virtual}} \Leftarrow x \ominus a$        // What was truly added - Rounded
$y \Leftarrow b \ominus b_{\text{virtual}}$        // round-off error
return $(x, y)$
```



```
FAST-TWO-SUM$(a, b)$
    $x \Leftarrow a \oplus b$
    $b_{\text{virtual}} \Leftarrow x \ominus a$
    $y \Leftarrow b \ominus b_{\text{virtual}}$
    return $(x, y)$
```



$$
\begin{aligned}
a + b &= x + y \\
&= a \oplus b + \ b \ominus b_{virtual}
\end{aligned}
$$

Fast TwoSum with result rounded up

Correct	Rounded	Really added	Correction
$a = 5081$	$a = 5081$	$x = 5175$	$b = 93.5$
$b = 93.5$	$b = 93.5$	$-a = -5081$	$-b_{virtual} = -94$
----------	----------	----------	----------
5174.5	$x = 5175$	$b_{virtual} = 94$	$y = -0.5$

$$(a + b) = (x + y)$$

$$5081 + 93.5 = (5175 - 0.5)$$

Fast TwoSum with result rounded down

Correct

$$\begin{array}{r} a = 5081 \\ b = \quad 93.4 \\ \hline 5174.4 \end{array}$$

Rounded

$$\begin{array}{r} a = 5081 \\ b = \quad 93.4 \\ \hline x = 5174 \end{array}$$

Really added

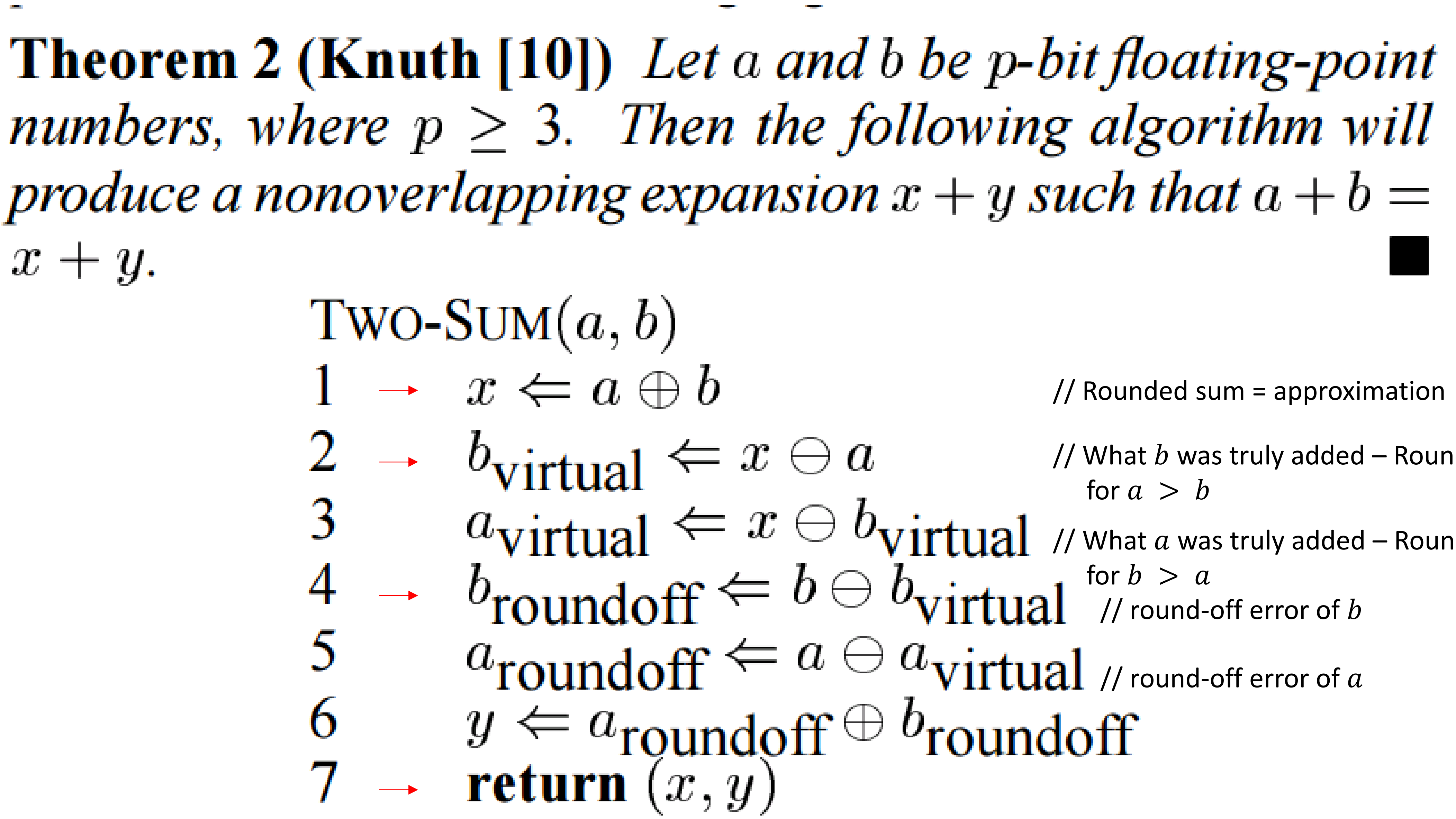
$$\begin{array}{r} x = \quad 5174 \\ -a = -5081 \\ \hline b_{virtual} = 93 \end{array}$$

Correction

$$\begin{array}{r} b = \quad 93.4 \\ -b_{virtual} = -94 \\ \hline y = \quad 0.4 \end{array}$$

$$(a + b) = (x + y)$$

$$5081 + 93.4 = (5174 + 0.4)$$



Theorem 2 (Knuth [10]) *Let a and b be p-bit floating-point numbers, where $p \geq 3$. Then the following algorithm will produce a nonoverlapping expansion $x + y$ such that $a + b = x + y$.* ■

```
TWO-SUM$(a, b)$
    $x \Leftarrow a \oplus b$                                              // Rounded sum = approximation
    $b_{\text{virtual}} \Leftarrow x \ominus a$                           // What $b$ was truly added – Rounded for $a > b$
    $a_{\text{virtual}} \Leftarrow x \ominus b_{\text{virtual}}$         // What $a$ was truly added – Rounded for $b > a$
    $b_{\text{roundoff}} \Leftarrow b \ominus b_{\text{virtual}}$        // round-off error of $b$
    $a_{\text{roundoff}} \Leftarrow a \ominus a_{\text{virtual}}$        // round-off error of $a$
    $y \Leftarrow a_{\text{roundoff}} \oplus b_{\text{roundoff}}$
    return $(x, y)$
```

Sum of two expansions (4-bit arithmetic)

Input: 1111+0.1001 and 1100 + 0.1

Output: 11100 + 0 + 0.0001

Zeroes slow down the computation – removed afterwards

Merge both input expansions into a single sequence *g* respecting the order of magnitudes

1111+ 1100 + 0.1001 + 0.1

Use LINEAR-EXPANSION-SUM (*g*)

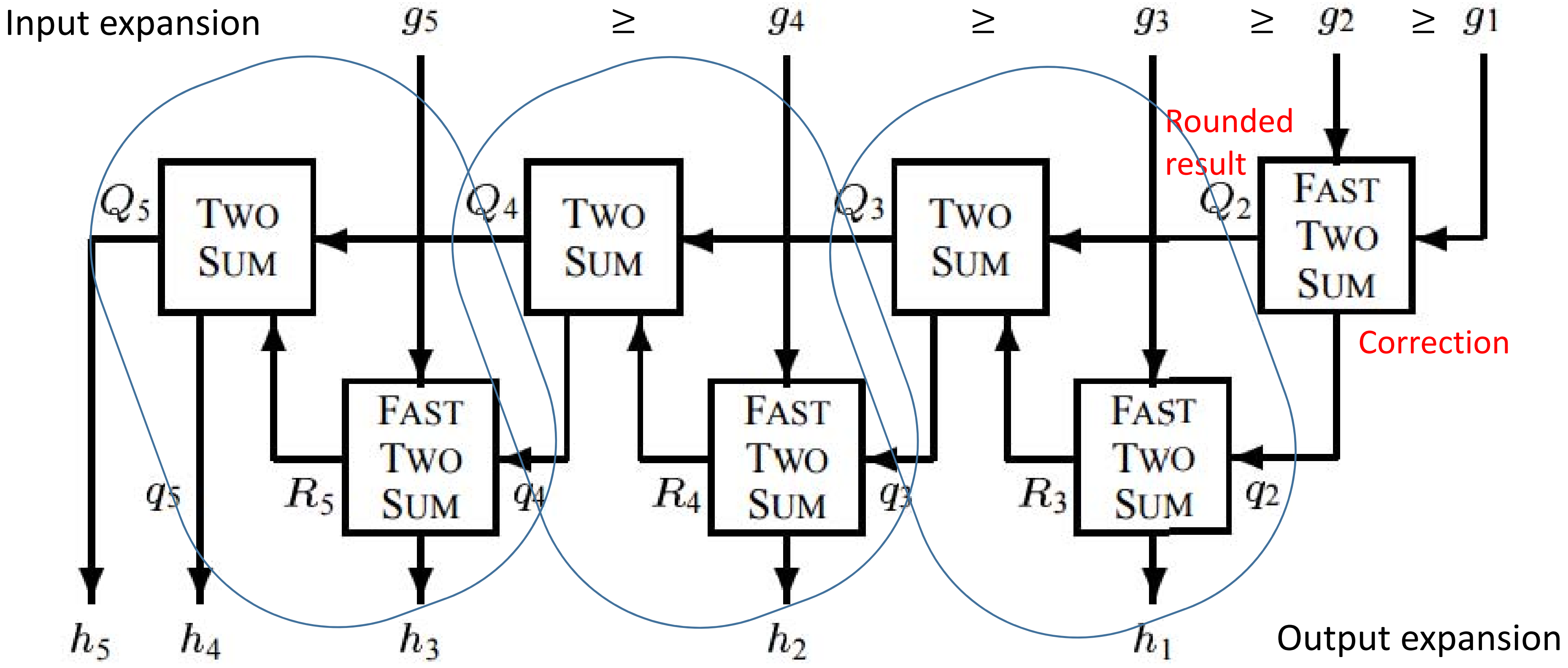


Figure 1: Operation of LINEAR-EXPANSION-SUM. The expansions g and h are illustrated with their most significant components on the left. $Q_i + q_i$ maintains an approximate running total. The FAST-TWO-SUM operations in the bottom row exist to clip a high-order bit off each q_i term, if necessary, before outputting it.

Multiplication

Multiplies two p-bit values a and b

1. Split both p-bit values into two halve (with ~p/2 bits)
2. perform four exact multiplications on these fragments.

$$a_{hi} \times a_{hi}, a_{hi} \times a_{lo}, a_{lo} \times a_{hi}, a_{lo} \times a_{lo},$$

The trick is to find a way to split a floating-point value in two.

SPLIT(a) operation

- Splits p bits into two non-overlapping halves ($\left\lfloor \frac{p}{2} \right\rfloor$ bits a_{hi} and $\left\lceil \frac{p}{2} \right\rceil - 1$ bits a_{lo})
- Missing bit is hidden in the signum of a_{lo}
- Example

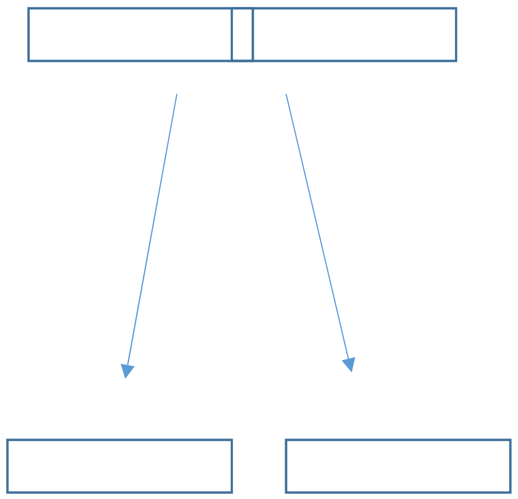
7bit number splits to two 3 bit significands

1001001 splits to 1010000 (101×2^4) and -111

73 = 80 - 7

Theorem 4 (Dekker [4]) *Let* a *be a* p*-bit floating-point number, where* $p \geq 3$. *The following algorithm will produce a* $\lfloor \frac{p}{2} \rfloor$*-bit value* a_{hi} *and a nonoverlapping* $(\lceil \frac{p}{2} \rceil - 1)$*-bit value* a_{lo} *such that* $|a_{\text{hi}}| \geq |a_{\text{lo}}|$ *and* $a = a_{\text{hi}} + a_{\text{lo}}$. ∎

```
SPLIT($a$)
    $c \Leftarrow (2^{\lceil p/2 \rceil} + 1) \otimes a$
    $a_{\text{big}} \Leftarrow c \ominus a$
    $a_{\text{hi}} \Leftarrow c \ominus a_{\text{big}}$
    $a_{\text{lo}} \Leftarrow a \ominus a_{\text{hi}}$
    return $(a_{\text{hi}}, a_{\text{lo}})$
```



Theorem 5 (Veltkamp) *Let a and b be p-bit floating-point numbers, where $p \geq 4$. The following algorithm will produce a nonoverlapping expansion $x + y$ such that $ab = x + y$.* ■

```
Two-Product($a, b$)
    $x \Leftarrow a \otimes b$
    $(a_{\mathrm{hi}}, a_{\mathrm{lo}}) = \textsc{Split}(a)$
    $(b_{\mathrm{hi}}, b_{\mathrm{lo}}) = \textsc{Split}(b)$
    $err_1 \Leftarrow x \ominus (a_{\mathrm{hi}} \otimes b_{\mathrm{hi}})$
    $err_2 \Leftarrow err_1 \ominus (a_{\mathrm{lo}} \otimes b_{\mathrm{hi}})$
    $err_3 \Leftarrow err_2 \ominus (a_{\mathrm{hi}} \otimes b_{\mathrm{lo}})$
    $y \Leftarrow (a_{\mathrm{lo}} \otimes b_{\mathrm{lo}}) \ominus err_3$
    return $(x, y)$
```

Demonstration of SPLIT splitting a five-bit number into two two-bit numbers

$$
\begin{array}{rcccccccccccccl}
a & = & & & & & & & 1 & 1 & 1 & 0 & 1 & \\
2^3 a & = & & & & 1 & 1 & 1 & 0 & 1 & & & & \times 2^3 \\
\cline{5-13}
c & = & (2^3+1) \otimes a & = & 1 & 0 & 0 & 0 & 0 & & & & & \times 2^4 \\
a & = & & & & & & & 1 & 1 & 1 & 0 & 1 & \\
\cline{5-13}
a_{\mathrm{big}} & = & c \ominus a & = & & 1 & 1 & 1 & 0 & 0 & & & & \times 2^3 \\
 & & & & & & & & & & & & & \\
a_{\mathrm{hi}} & = & c \ominus a_{\mathrm{big}} & = & & & & 1 & 0 & 0 & 0 & 0 & & \times 2^1 \\
 & & & & & & & & & & & & & \\
a_{\mathrm{lo}} & = & a \ominus a_{\mathrm{hi}} & = & & & & & & & - & 1 & 1 &
\end{array}
$$

Demonstration of TWO-PRODUCT in six-bit arithmetic

$$
\begin{array}{rcccccccccccccccl}
a & = & & & & & & & & & 1 & 1 & 1 & 0 & 1 & 1 & \\
b & = & & & & & & & & & 1 & 1 & 1 & 0 & 1 & 1 & \\
\cline{5-16}
x & = & a \otimes b & = & 1 & 1 & 0 & 1 & 1 & 0 & & & & & & & \times 2^6 \\
 & & a_{\mathrm{hi}} \otimes b_{\mathrm{hi}} & = & 1 & 1 & 0 & 0 & 0 & 1 & & & & & & & \times 2^6 \\
\cline{5-10}
err_1 & = & x \ominus (a_{\mathrm{hi}} \otimes b_{\mathrm{hi}}) & = & & & & 1 & 0 & 1 & 0 & 0 & 0 & & & & \times 2^3 \\
 & & a_{\mathrm{lo}} \otimes b_{\mathrm{hi}} & = & & & & & 1 & 0 & 1 & 0 & 1 & 0 & & & \times 2^2 \\
\cline{8-14}
err_2 & = & err_1 \ominus (a_{\mathrm{lo}} \otimes b_{\mathrm{hi}}) & = & & & & & 1 & 0 & 0 & 1 & 1 & 0 & & & \times 2^2 \\
 & & a_{\mathrm{hi}} \otimes b_{\mathrm{lo}} & = & & & & & 1 & 0 & 1 & 0 & 1 & 0 & & & \times 2^2 \\
\cline{9-14}
err_3 & = & err_2 \ominus (a_{\mathrm{hi}} \otimes b_{\mathrm{lo}}) & = & & & & & & & - & 1 & 0 & 0 & 0 & 0 & \\
 & & a_{\mathrm{lo}} \otimes b_{\mathrm{lo}} & = & & & & & & & & & 1 & 0 & 0 & 1 & \\
\cline{11-16}
-y & = & err_3 \ominus (a_{\mathrm{lo}} \otimes b_{\mathrm{lo}}) & = & & & & & & & - & 1 & 1 & 0 & 0 & 1 &
\end{array}
$$

The resulting expansion is $110110 \times 2^6 + 11001$

Adaptive arithmetic

- Expensive – avoid when possible
- Some applications need results with absolute error below a threshold
- Set of procedures with different precision (& speed) + error bounds
- For each input – compute the error bounds and choose the procedure

But

- Sometimes hard to determine error before computation
- Especially when relative error needed – like sign of expression comp.
  - Result can be much larger than error bound – exact arithmetic will suffice
  - Result can be near zero – must be evaluated exactly

Shewchuk predicates

- Compute a sequence of increasingly accurate results
- Testing each for accuracy
- Not using separate procedures BUT
- Using intermediate results as steps to more accurate results (work already done is not discarded, but refined)
- Idea: presented routines can be split to two parts
  - Line 1 gives an approximate result - run each time
  - Remaining lines compute the roundoff error – delayed until needed, if ever …

Principle of adaptive computation

Distance of two points $(b_x - a_x)^2 + \left(b_y - a_y\right)^2$

Store $b_x - a_x$ as $x_1 + y_1$

and $b_y - a_y$ as $x_2 + y_2$

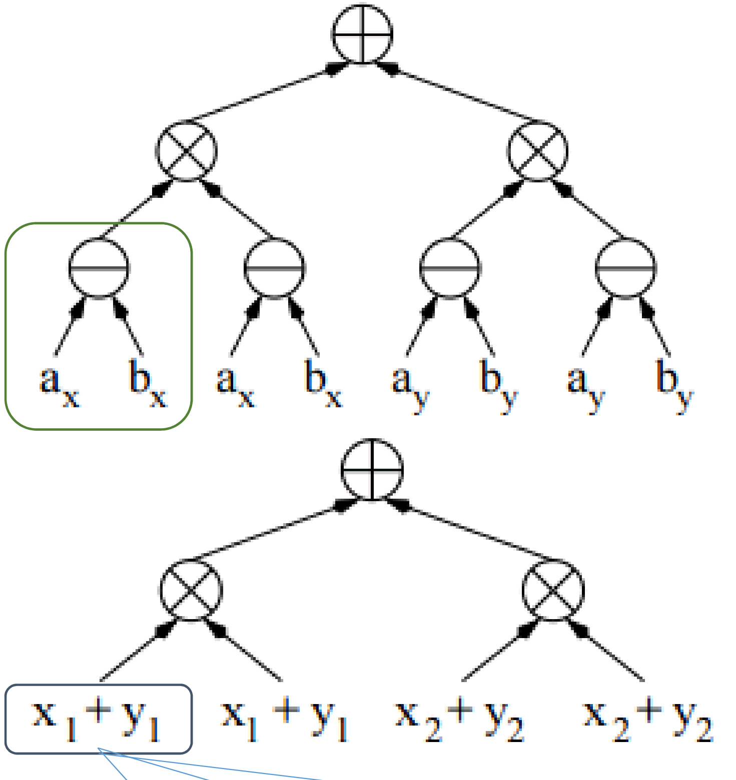
$$(x_1^2 + 2x_1y_1 + y_1^2) + (x_2^2 + 2x_2y_2 + y_2^2)$$

Reorder terms according to their size

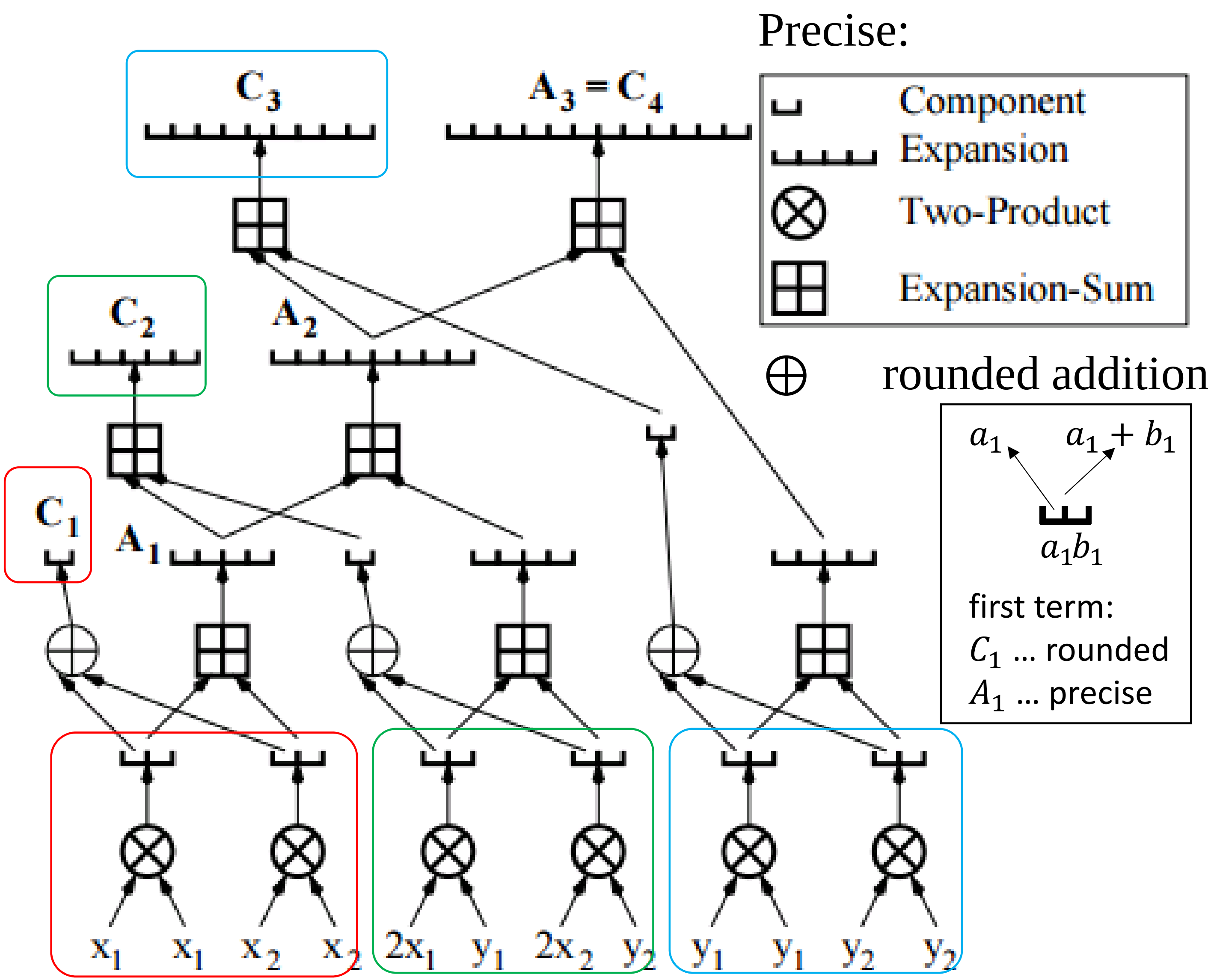
$$(x_1^2 + x_2^2) + (2x_1y_1 + 2x_2y_2) + (y_1^2 + y_2^2)$$

Compute them only if needed

$$(b_x - a_x)^2 + (b_y - a_y)^2$$



$$(\boxed{x_1^2} + \boxed{2x_1y_1} + \boxed{y_1^2}) +$$
$$(\boxed{x_2^2} + \boxed{2x_2y_2} + \boxed{y_2^2})$$



first term:
C_1 ... rounded
A_1 ... precise

pivot p

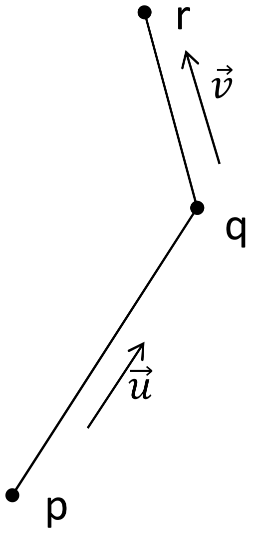
Orientation predicate - definition

$$\text{orientation}(p,q,r) = \text{sign}\left(\det\begin{bmatrix}1 & p_x & p_y\\ 1 & q_x & q_y\\ 1 & r_x & r_y\end{bmatrix}\right) =$$

$$= \text{sign}\Big((q_x - p_x)\big(r_y - p_y\big) - \big(q_y - p_y\big)(r_x - p_x)\Big),$$

$$\text{where point } p = \big(p_x, p_y\big), \ldots$$

$$= \text{third coordinate of} = (\vec{u} \times \vec{v}),$$



Three points	$\text{orientation}(p,q,r) =$
lie on common line	= 0
form a left turn	= +1 (positive)
form a right turn	= –1 (negative)

pivot r

Experiment with orientation predicate

- orientation(p,q,r) = sign($(p_x-r_x)(q_y-r_y)-(p_y-r_y)(q_x-r_x)$)

Ideal return values