ALG 08

Merge sort

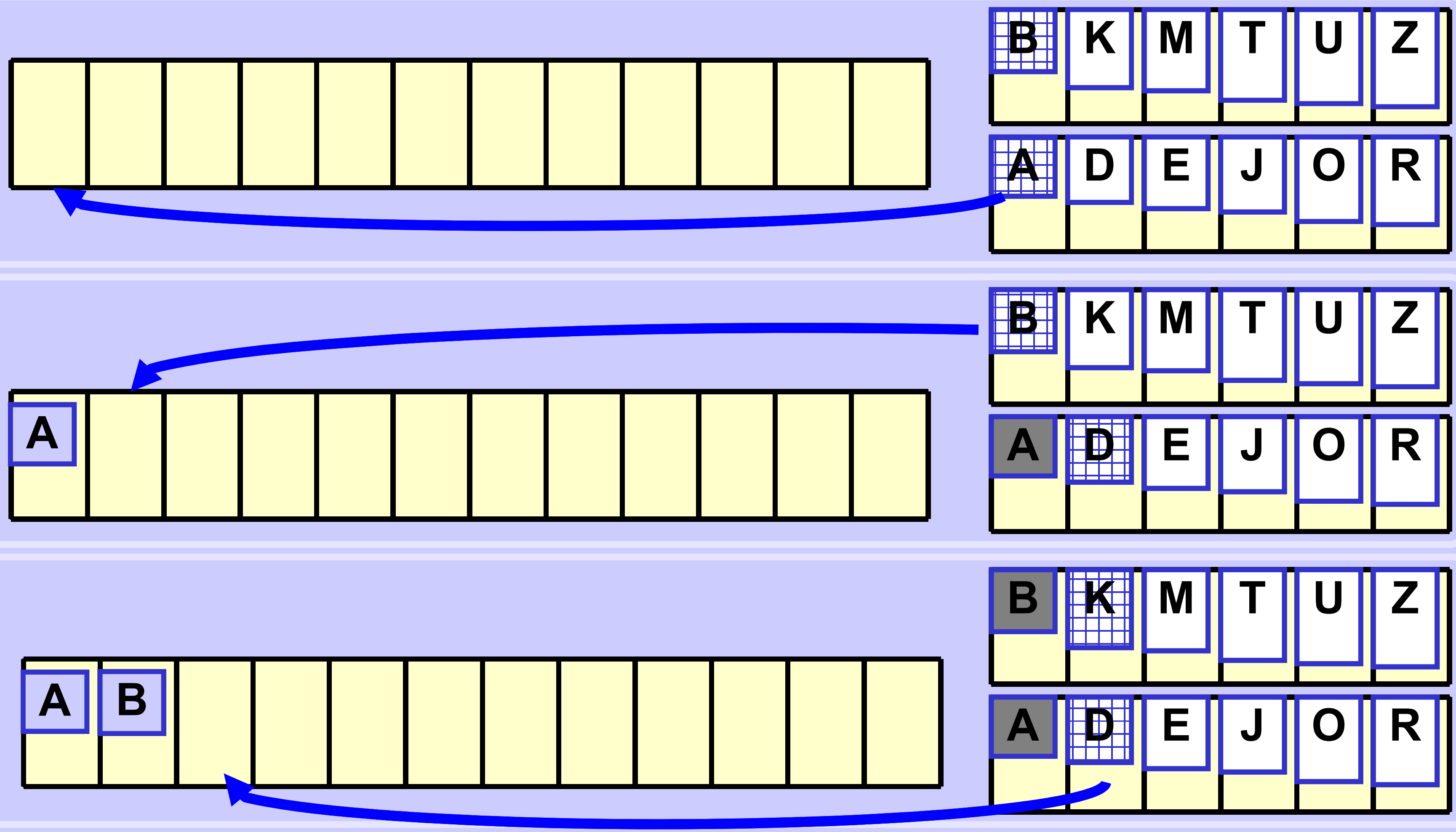
Heap Sort

Priority queue implemented with binary heap

Merge sort

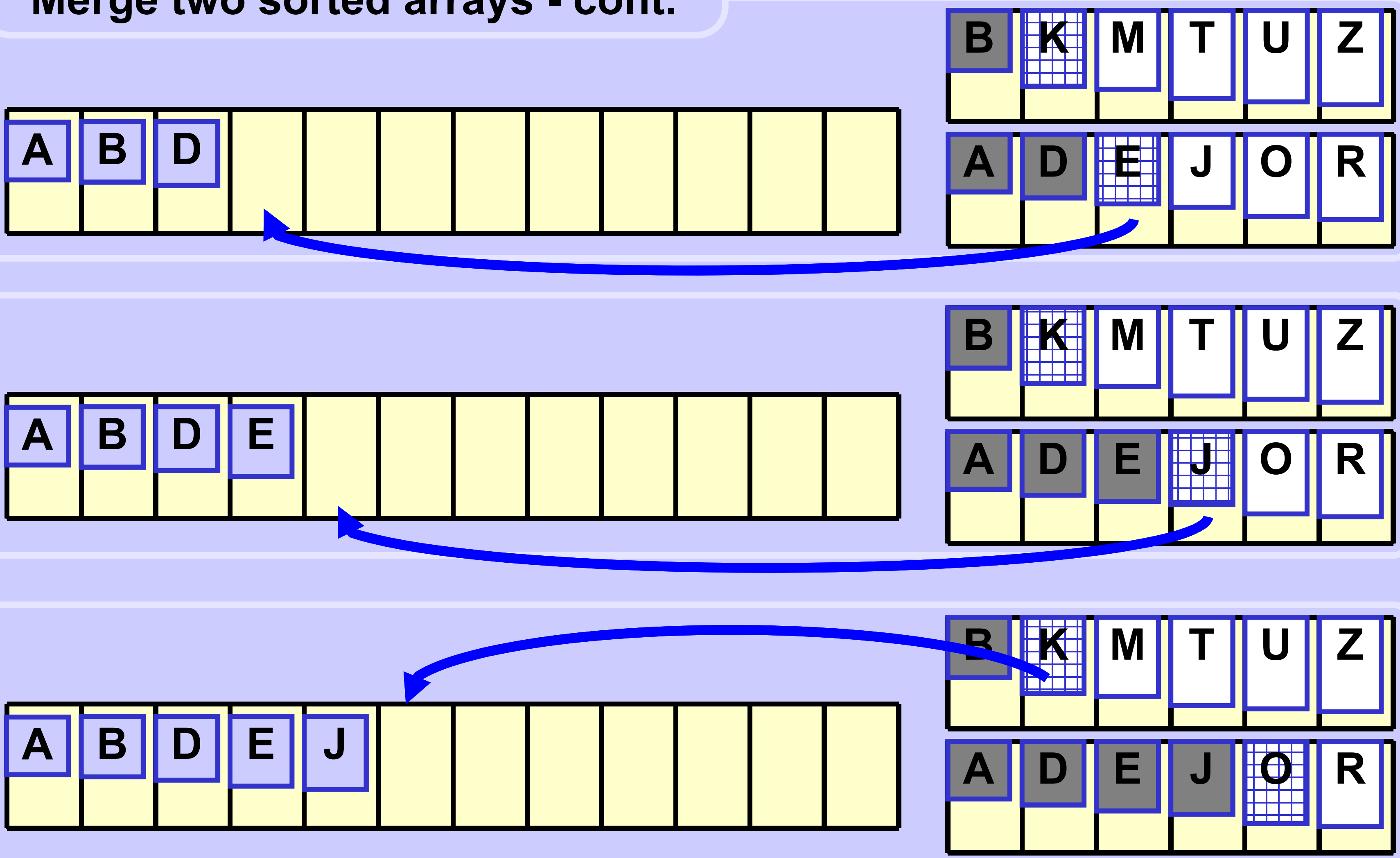
Merge two sorted arrays

Compared elements



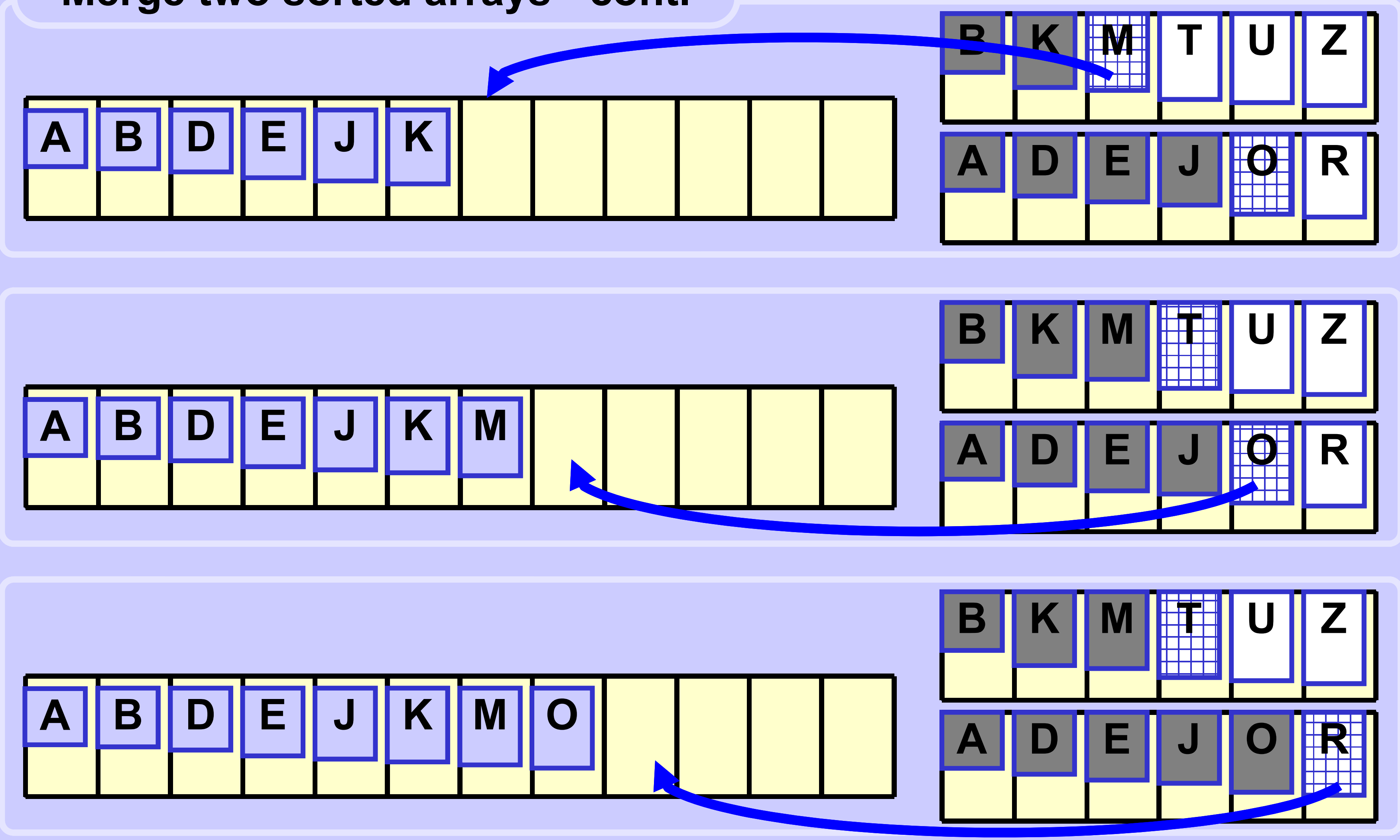
Merge sort

Merge two sorted arrays - cont.



Merge sort

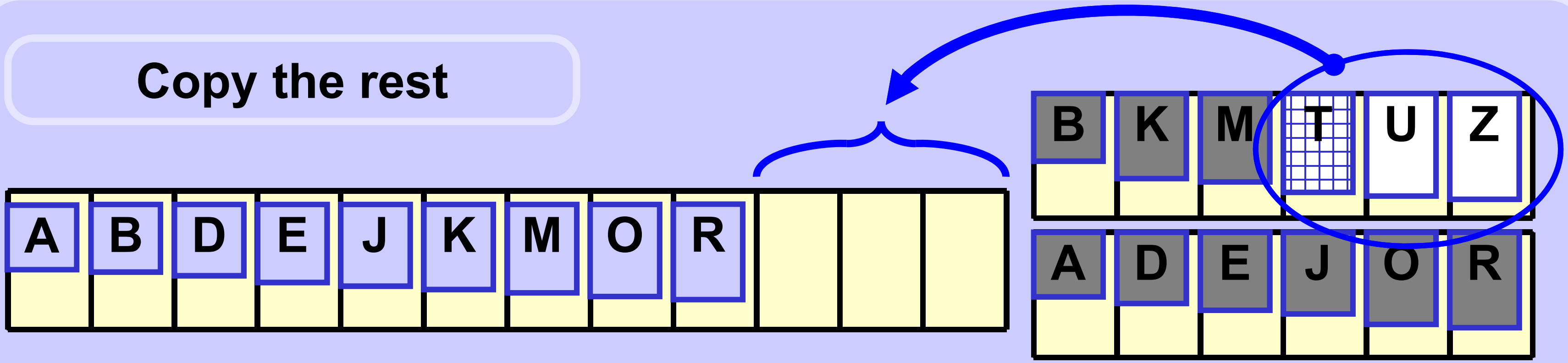
Merge two sorted arrays - cont.



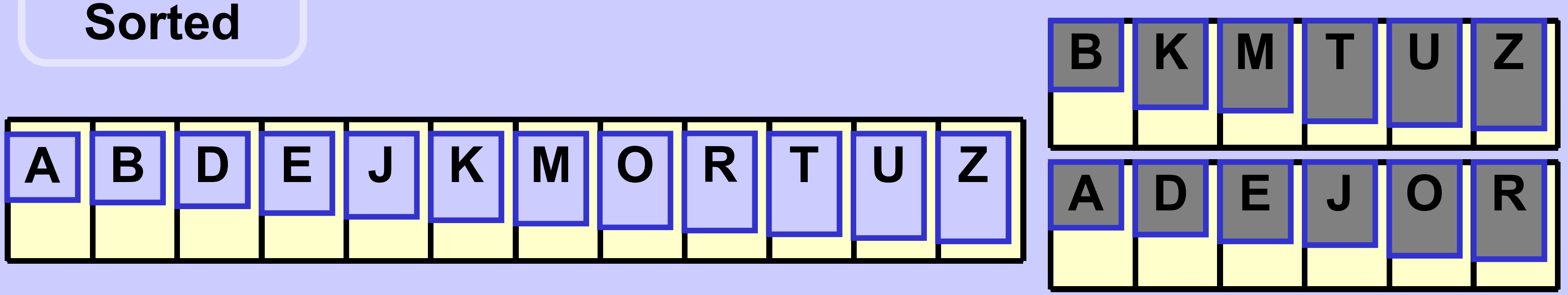
Merge sort

Merge two sorted arrays - cont.

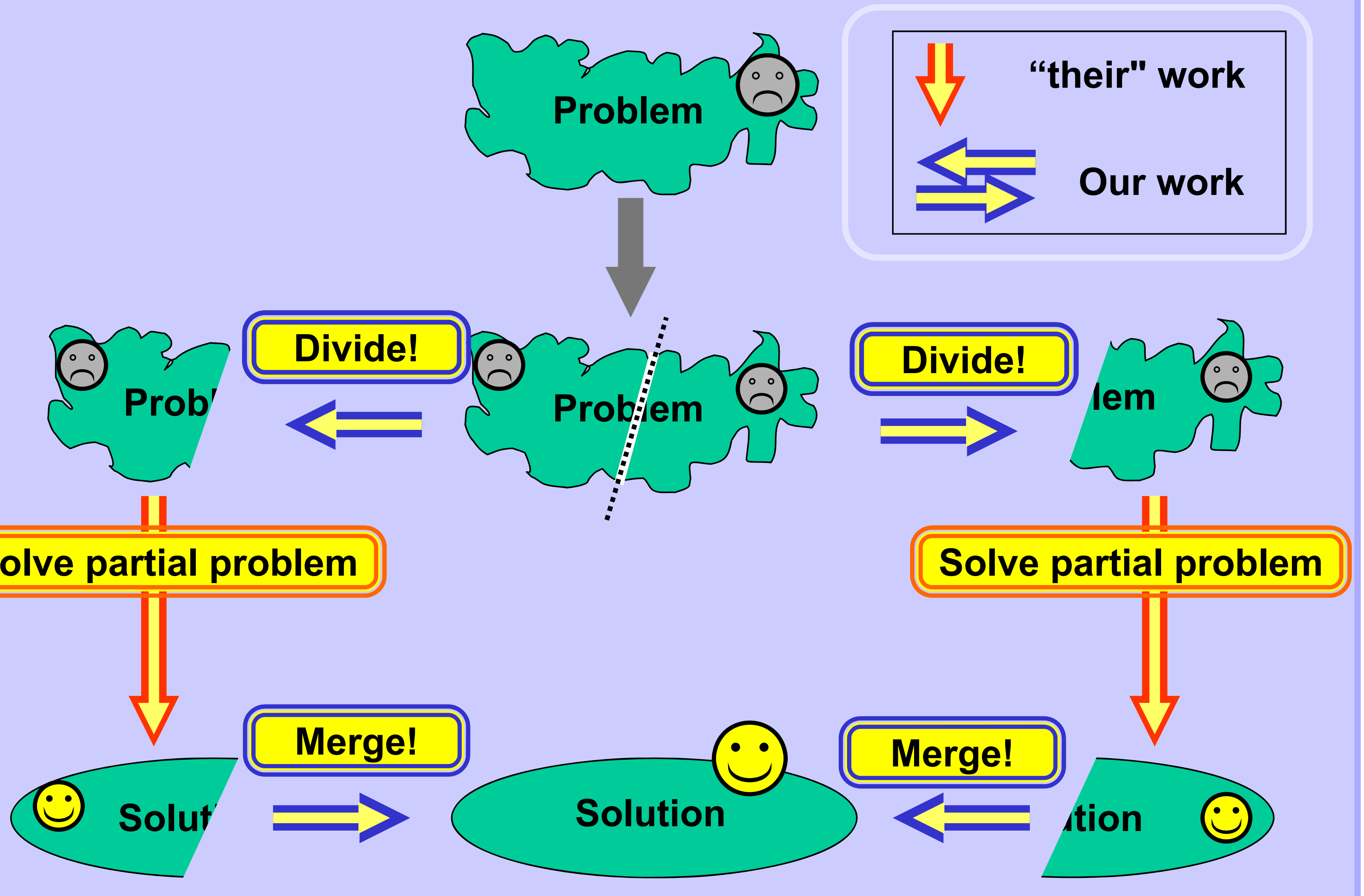
Copy the rest



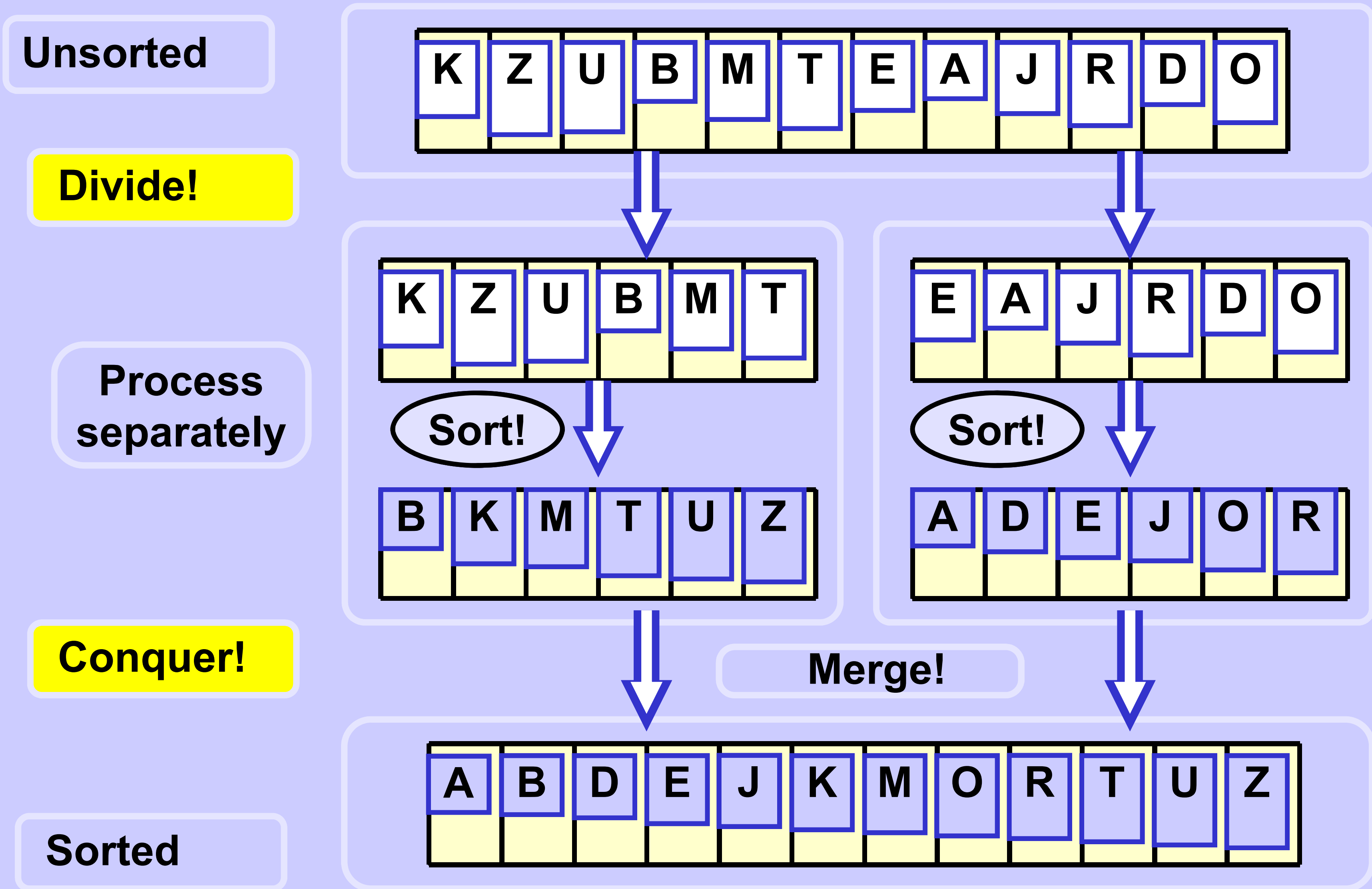
Sorted



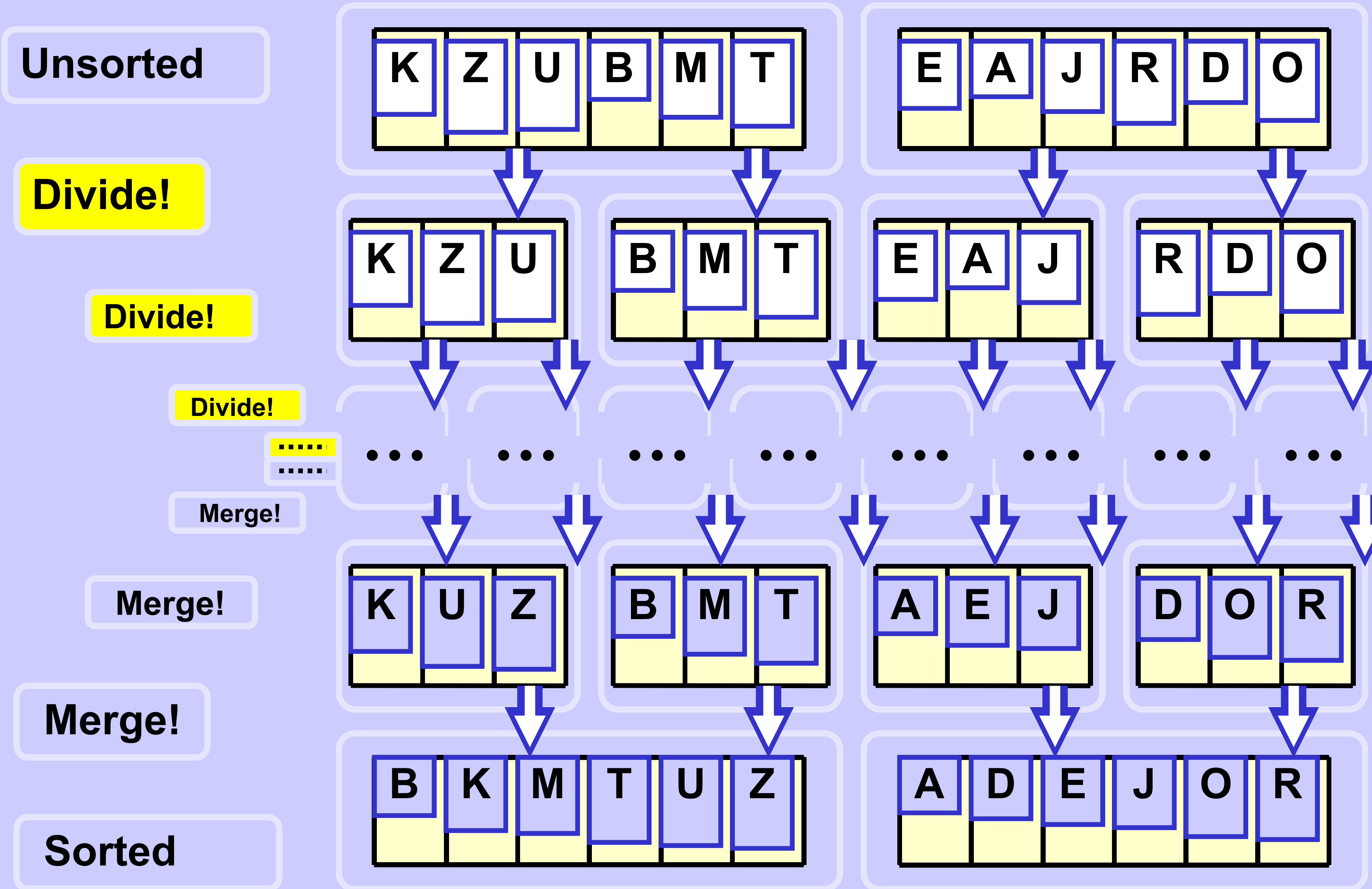
Divide and conquer! Divide et impera!



Merge sort


Unsorted
K Z U B M T E A J R D O
Divide!
K Z U B M T
E A J R D O
Process separately
Sort!
Sort!
B K M T U Z
A D E J O R
Conquer!
Merge!
A B D E J K M O R T U Z
Sorted

Merge sort


Unsorted
Divide!
Divide!
Divide!
Merge!
Merge!
Merge!
Sorted
K Z U B M T
E A J R D O
K Z U
B M T
E A J
R D O
K U Z
B M T
A E J
D O R
B K M T U Z
A D E J O R

Merge sort

```
def merge (inArr, outArr, low, high):
    half = (low+high) // 2
    i1 = low
    i2 = half + 1
    j = low;
    # compare and merge
    while i1 <= half and i2 <= high:
        if inArr[i1] <= inArr[i2]:
            outArr[j] = inArr[i1]; i1 += 1
        else:
            outArr[j] = inArr[i2]; i2 += 1
        j += 1
    # copy the rest
    while i1 <= half:
        outArr[j] = inArr[i1]; i1 += 1; j += 1
    while i2 <= high:
        outArr[j] = inArr[i2]; i2 += 1; j += 1
```

Merge sort

```
def _mergeSort (arr, auxArr, low, high):

  if low >= high: return   # too small!
  half = (low+high) // 2

  # sort to auxArr
  _mergeSort(arr, auxArr, low, half)     # left half
  _mergeSort(arr, auxArr, half+1, high) # right half
  merge(arr, auxArr, low, high)

  # copy back from auxArr
  for i in range(low, high+1):
    arr[i] = auxArr[i]
```

Merge sort - improved use of auxArr

```
def _mergeSortX (arr, auxArr, low, high, depth):

  if low >= high: return   # too small!
  half = (low+high) // 2

  _mergeSortX(arr, auxArr, low,    half, depth+1)
  _mergeSortX(arr, auxArr, half+1, high, depth+1)

  # note the swaping of arr and auxArr
  if depth%2 == 0: merge(auxArr, arr, low, high)
  else:            merge(arr, auxArr, low, high)


def mergeSortX (arr):
  auxArr = arr[:]  # auxArr = copy(arr)
  _mergeSortX(arr, auxArr, 0, len(arr)-1, 0)
```

Merge sort

Asymptotic complexity

Divide! .... $\log_2(n)$ times $\Rightarrow$

$\Rightarrow$ Merge! .... $\log_2(n)$ times

Divide! .... $\Theta(1)$ operations

Merge! .... $\Theta(n)$ operations

Total ...... $\Theta(n) \cdot \Theta(\log_2(n)) = \Theta(n \cdot \log_2(n))$ operations

Asymptotic complexity of Merge sort is $\Theta(n \cdot \log_2(n))$

Merge sort

Stability

Divide! .... **Does not move the elements.**

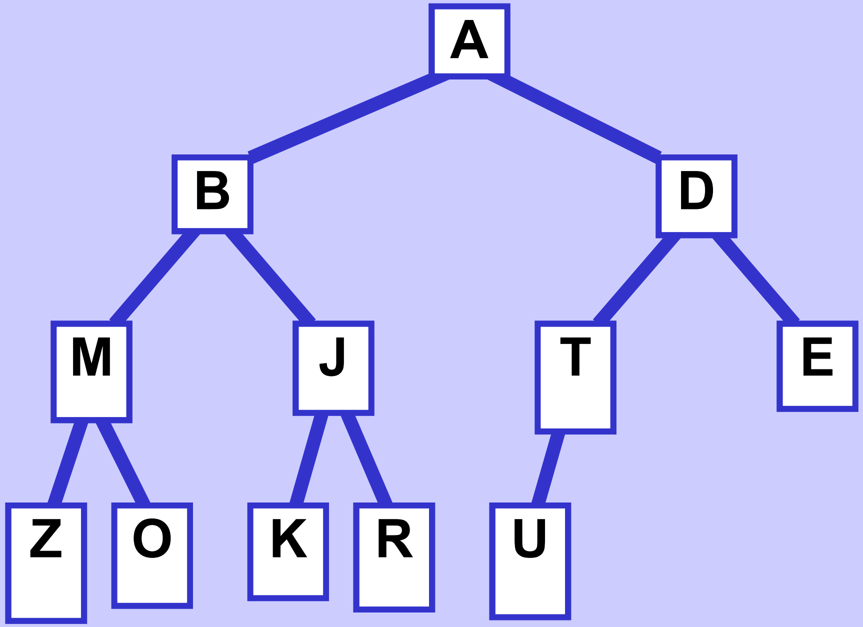
Merge! .... **“ if (in[i1] <= in[i2]) { out[j] = in[i1]; …”**

When the two compared and merged elements are equal, merge the left one first.

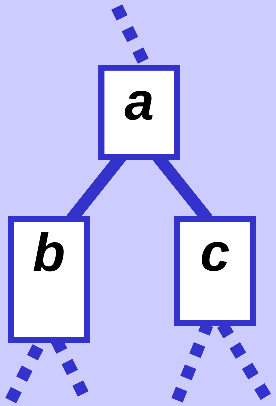
Merge sort is stable.

Heap sort

Heap



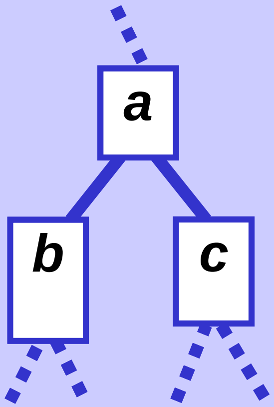
Heap property.



$a \leq b$ && $a \leq c$

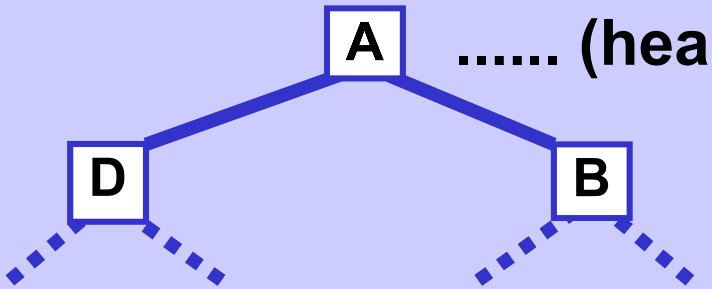
Heap sort

Terminology



a ...... predecessor, parent of *b* *c*

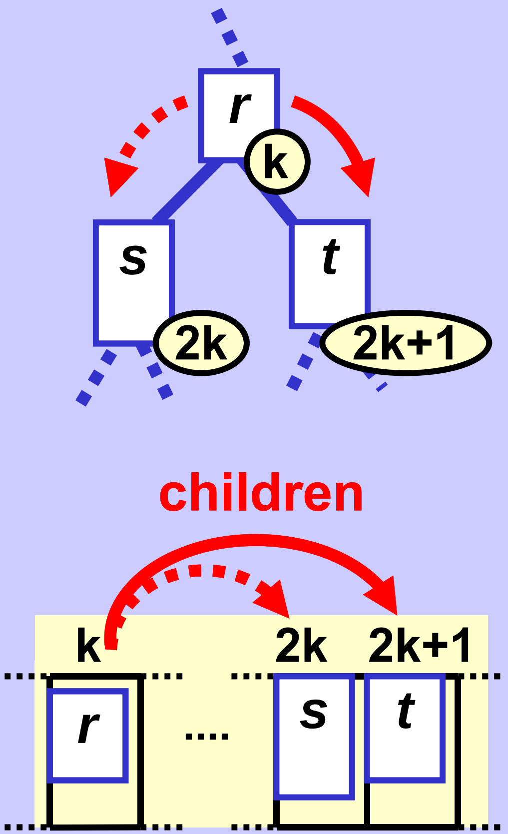
b , *c* ...... successor, child of *a*

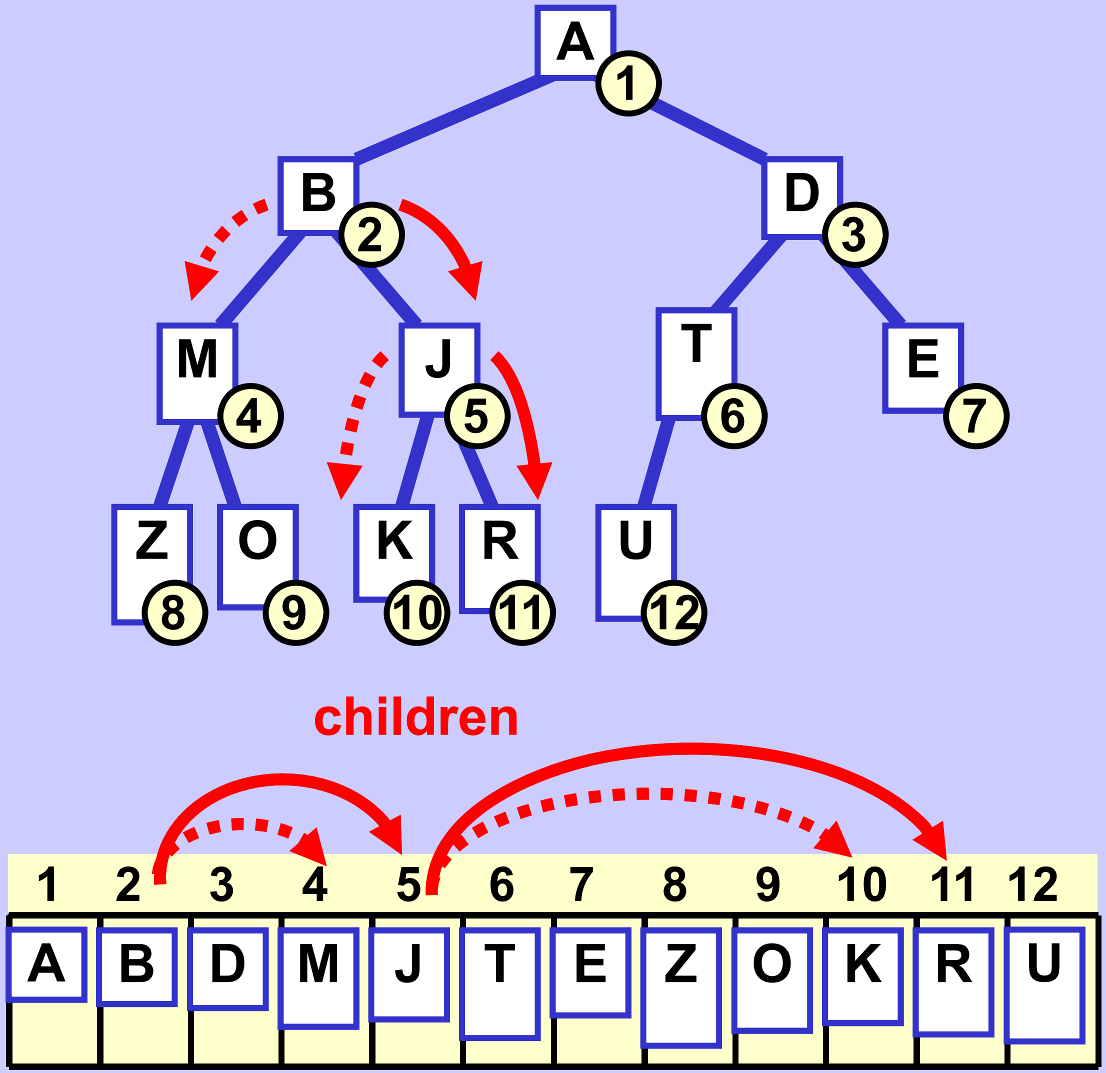


...... (heap) top

Heap sort

Heap stored in an array

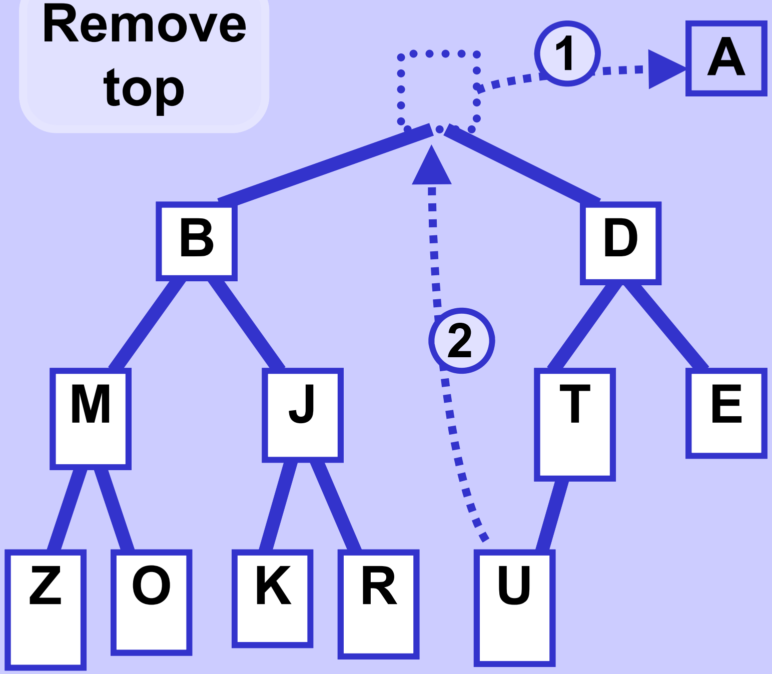




Heap repair

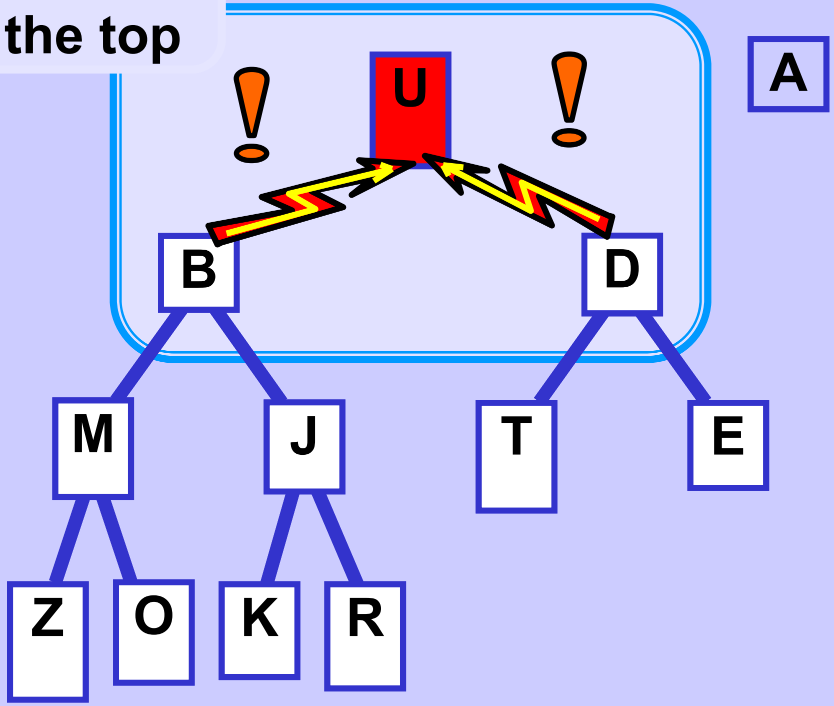
Top removed (1)

① Remove top



② last ⟶ first

③ Put at the top



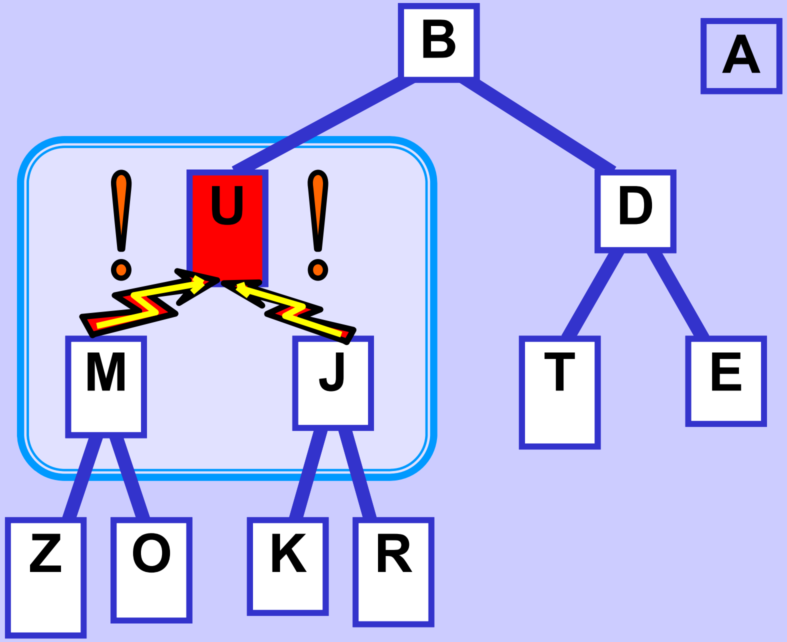
U > B, U > D, <u>B < D</u>

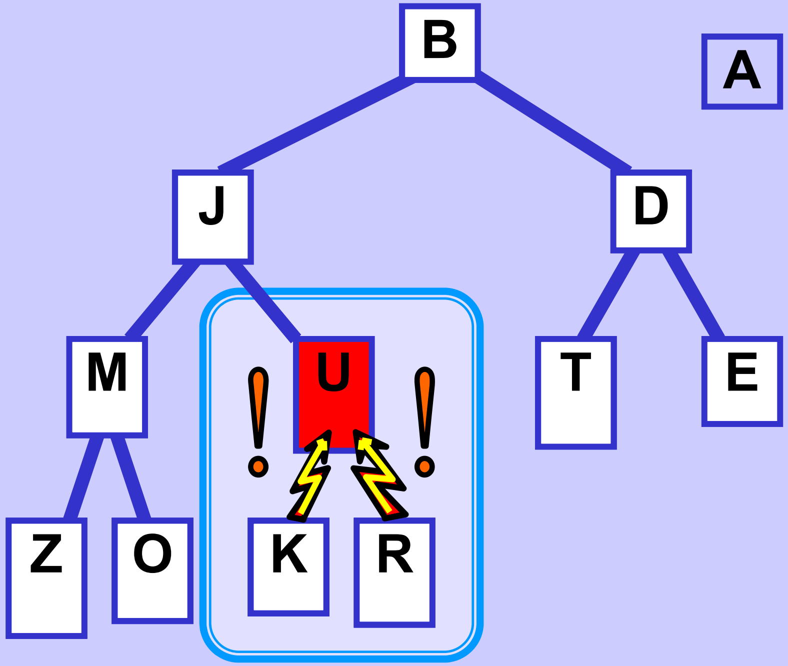
⇒ swap B ⟷ U

Heap repair

Top removed (2)

③ Put at the top - cont...





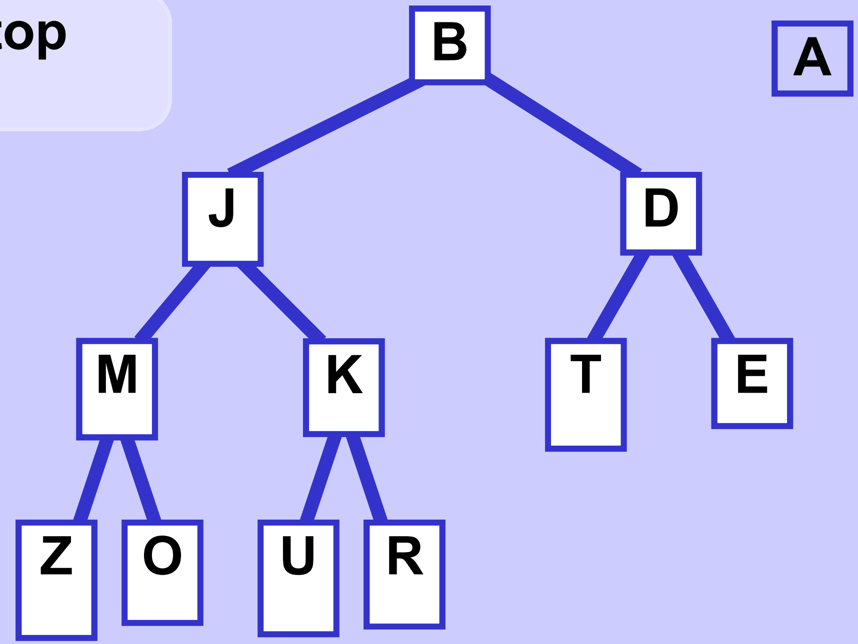
U > M, U > J, J < M
⇒ swap J ⟷ U

U > K, U > R, K < R
⇒ swap K ⟷ U

Heap repair

Top removed (3)

③ Put at the top
- done.

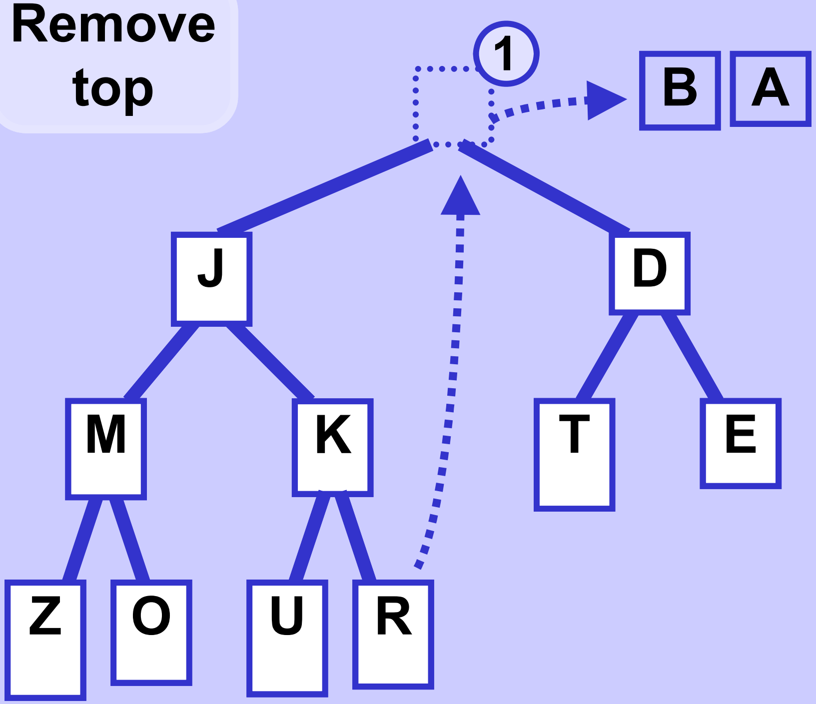


New heap

Heap repair

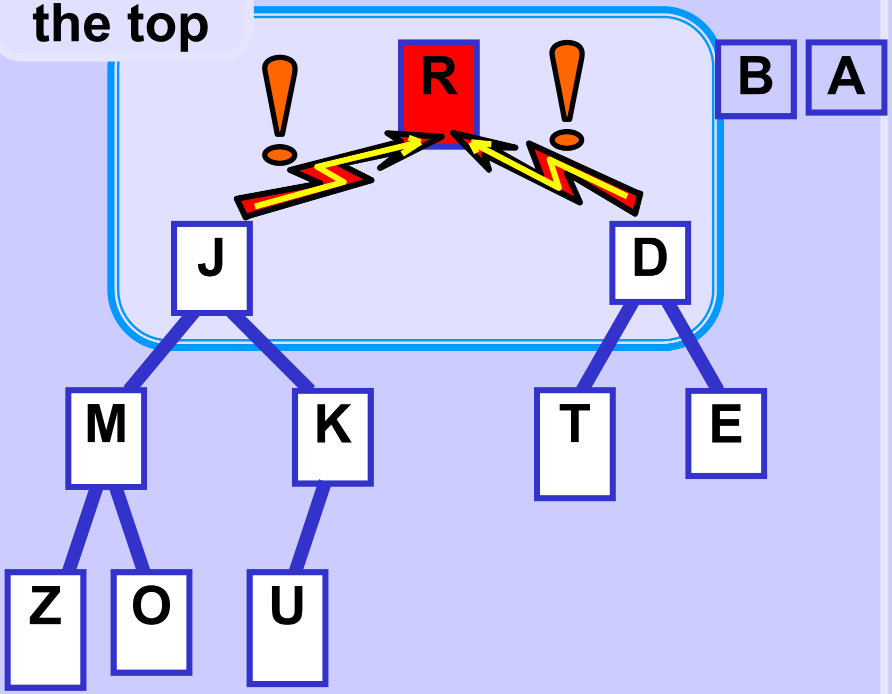
Top removed II (1)

① Remove top



② last ⟶ first

③ Put at the top



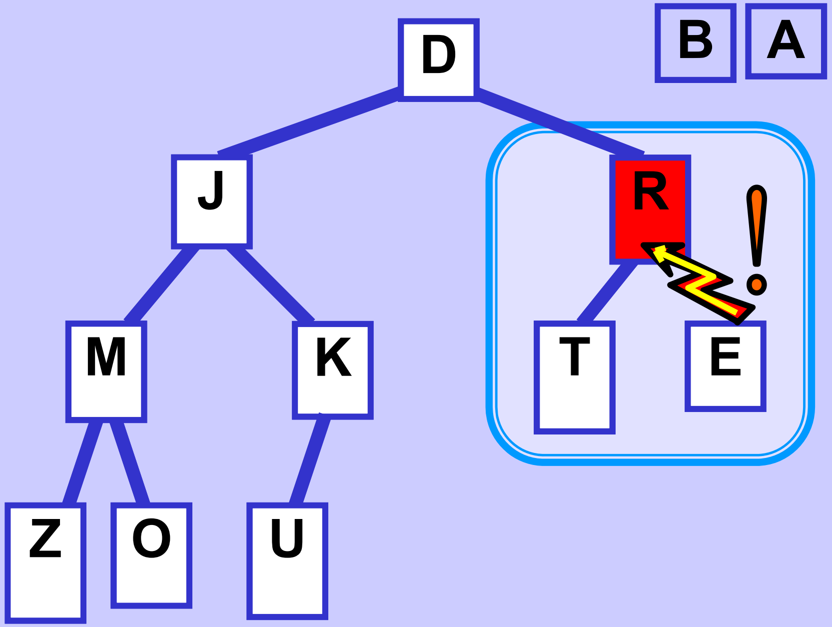
R > J, R > D, <u>D < J</u>

⇒ swap D ⟷ R

Heap repair

Top removed II (2)

③ Put at the top - cont.

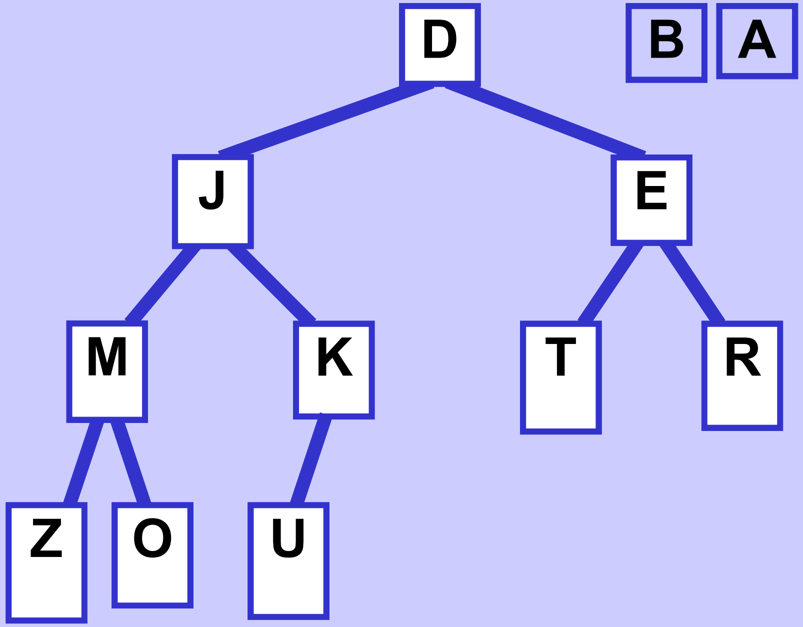


R < T, R > E

⇒ swap E ⟷ R

Top removed II (3)

③ Put at the top - done.



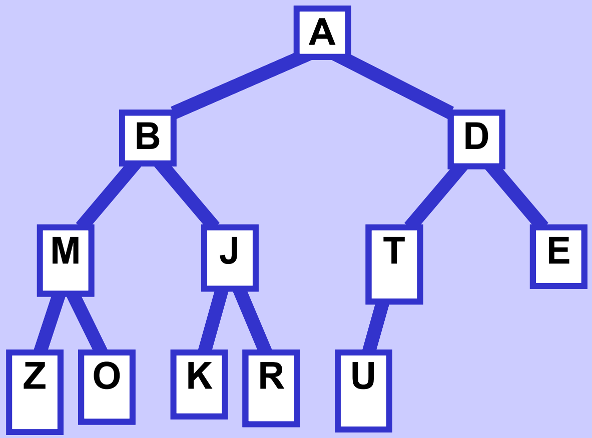
New heap

Heap sort

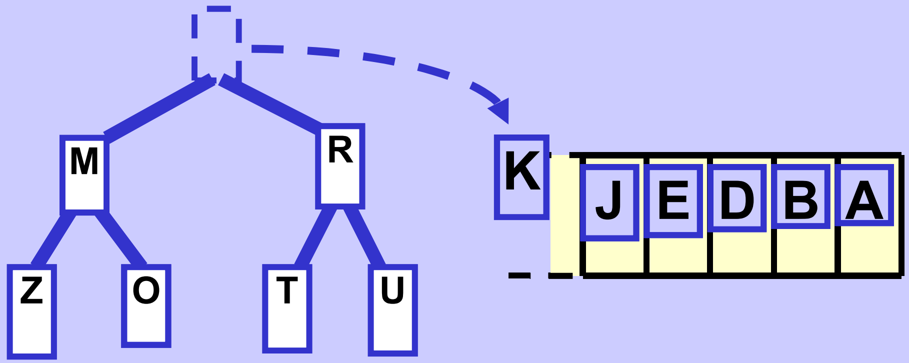
I Unsorted

R	J	U	Z	B	T	E	M	O	K	A	D

II Make a heap



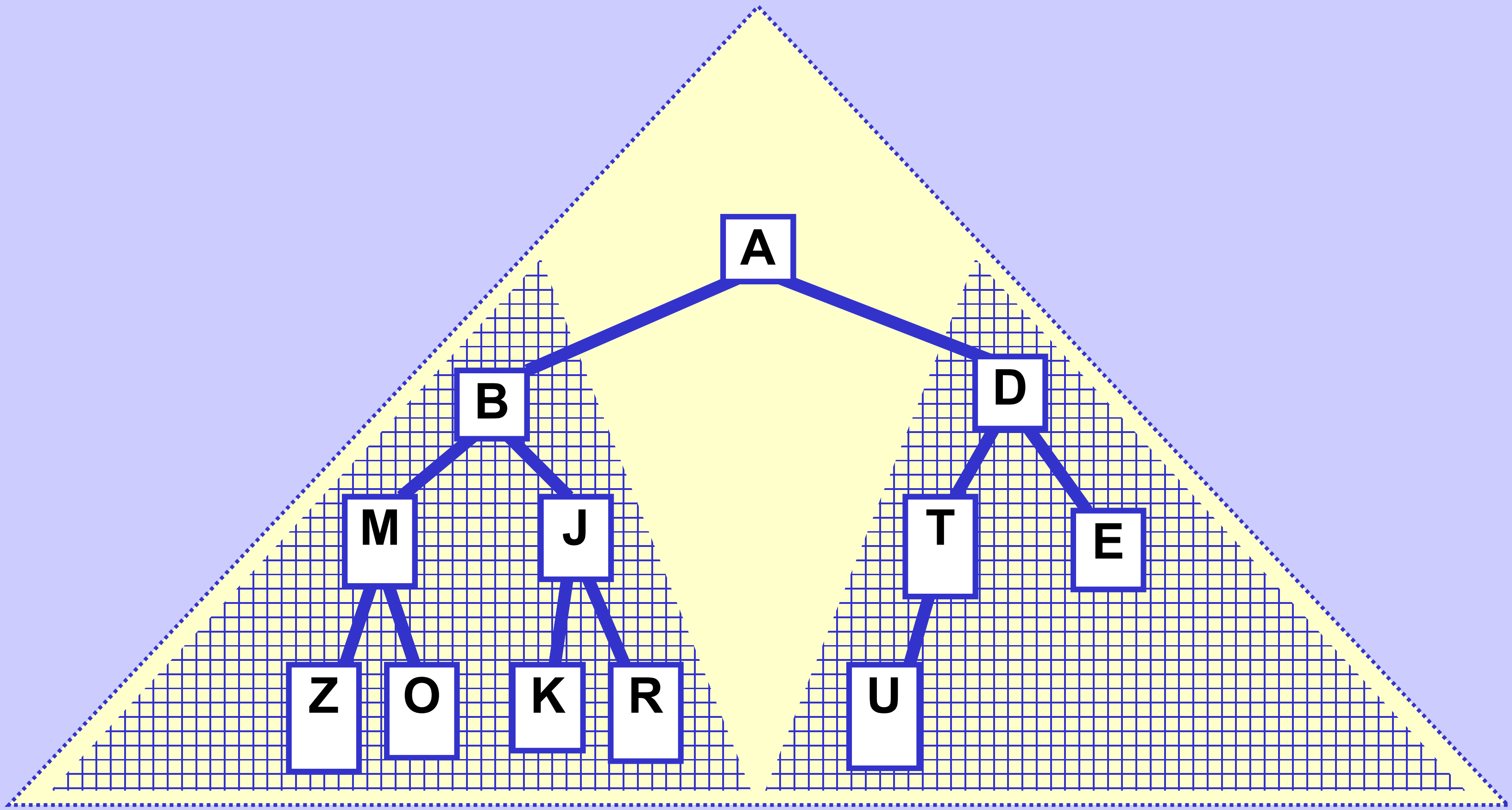
III for (i = 0; i < n; i++)
a[i] = “remove top”;

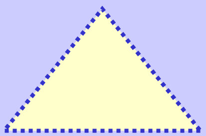
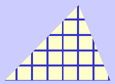
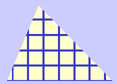


IV Sorted

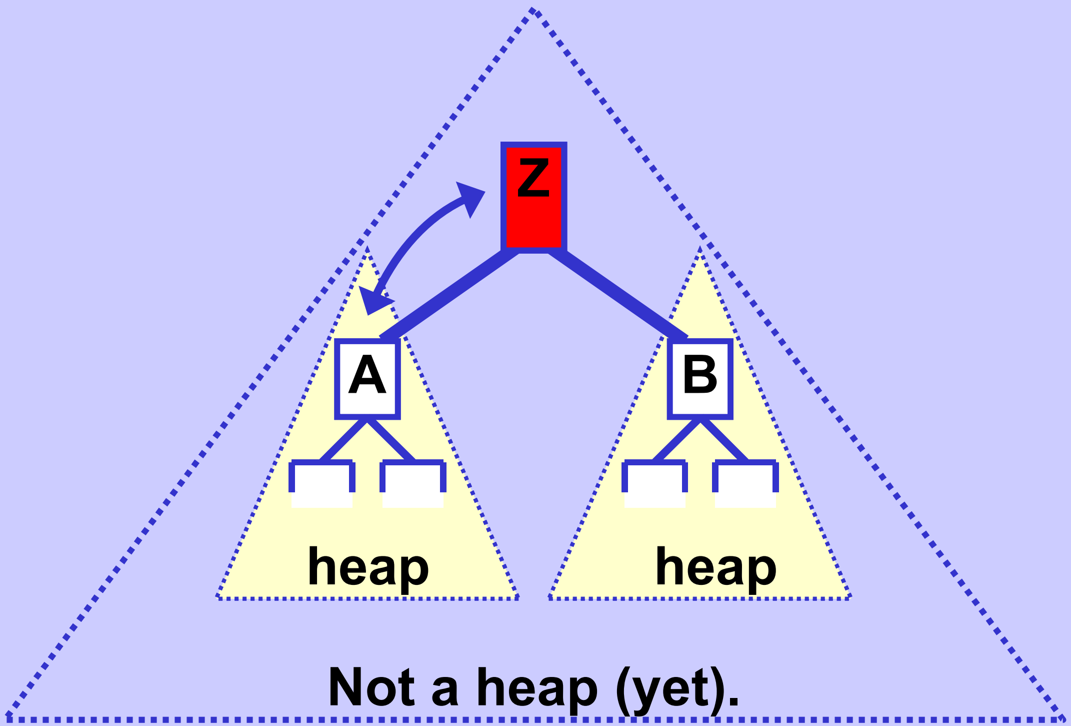
Z	U	T	R	O	M	K	J	E	D	B	A

Recursive property of "Being a heap"



 Is a heap ⇒  is a heap and  is a heap.

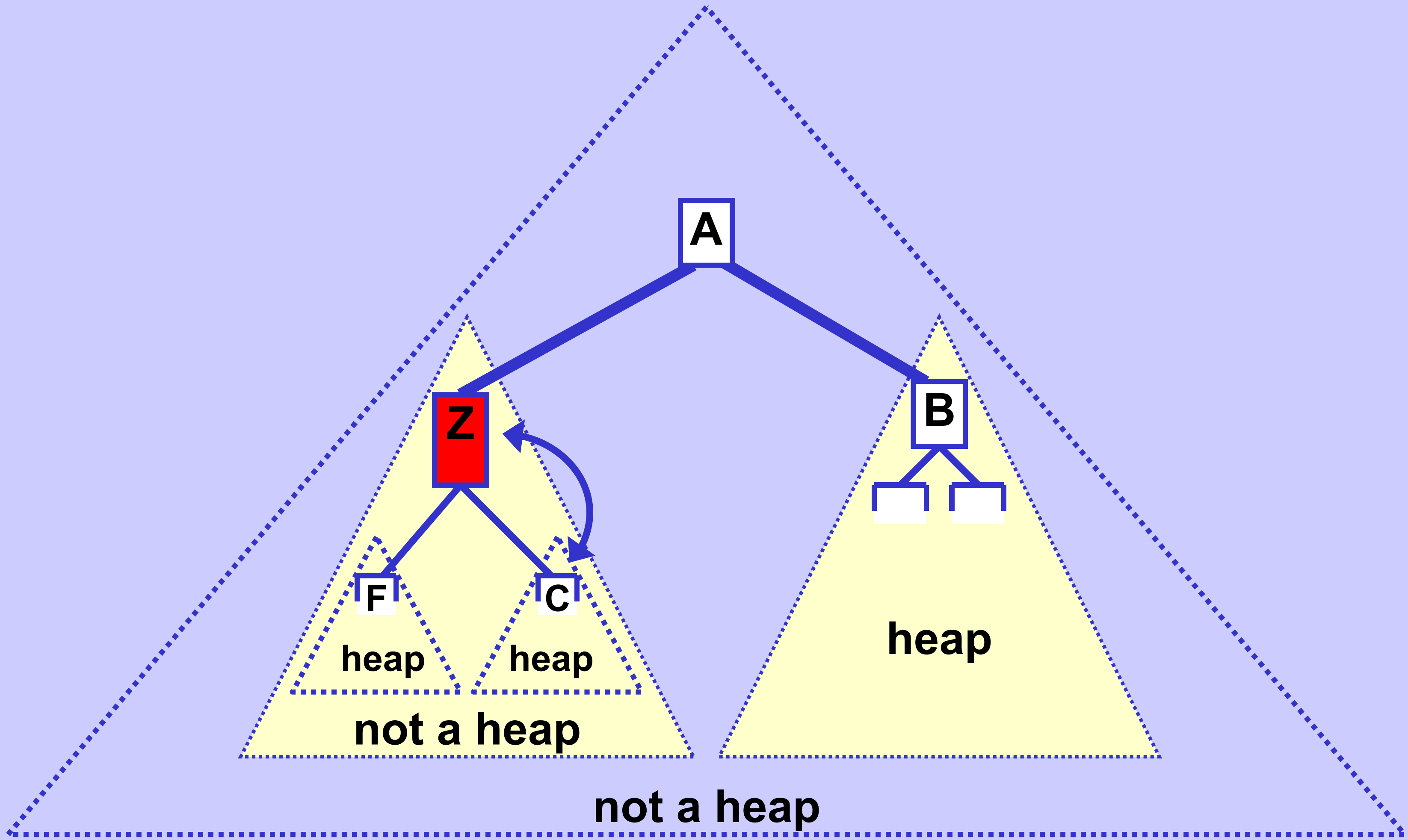
Make one bigger heap from two smaller ones



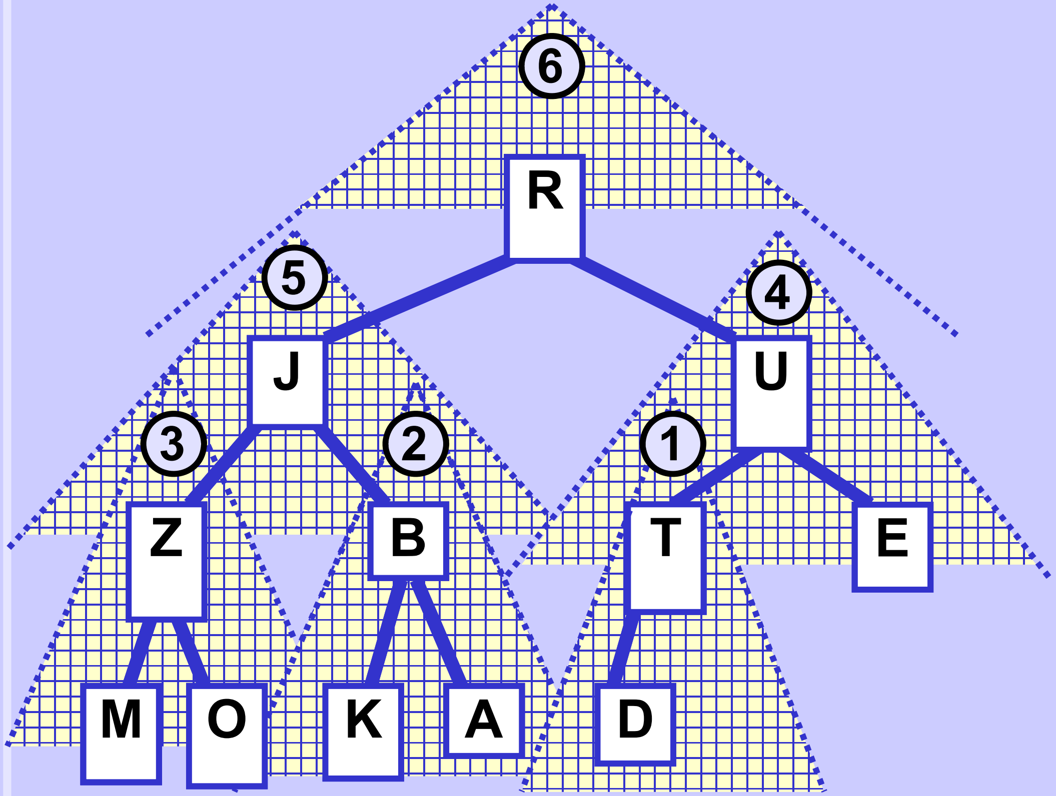
Z > A or Z > B

$\Rightarrow$ swap: Z $\leftrightarrow$ min(A,B)

Make one bigger heap from two smaller ones



Create a heap



Make a heap in ① ...

... make a heap in ② ...

... make a heap in ③ ...

... make a heap in ④ ...

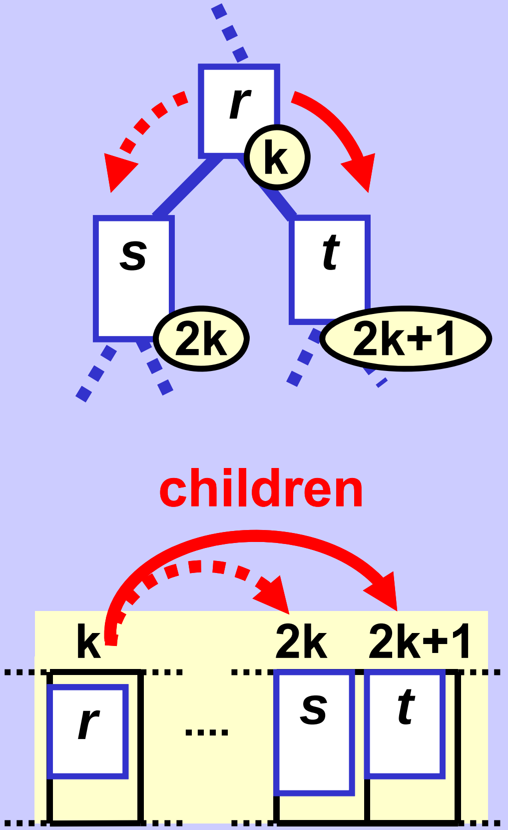
... make a heap in ⑤ ...

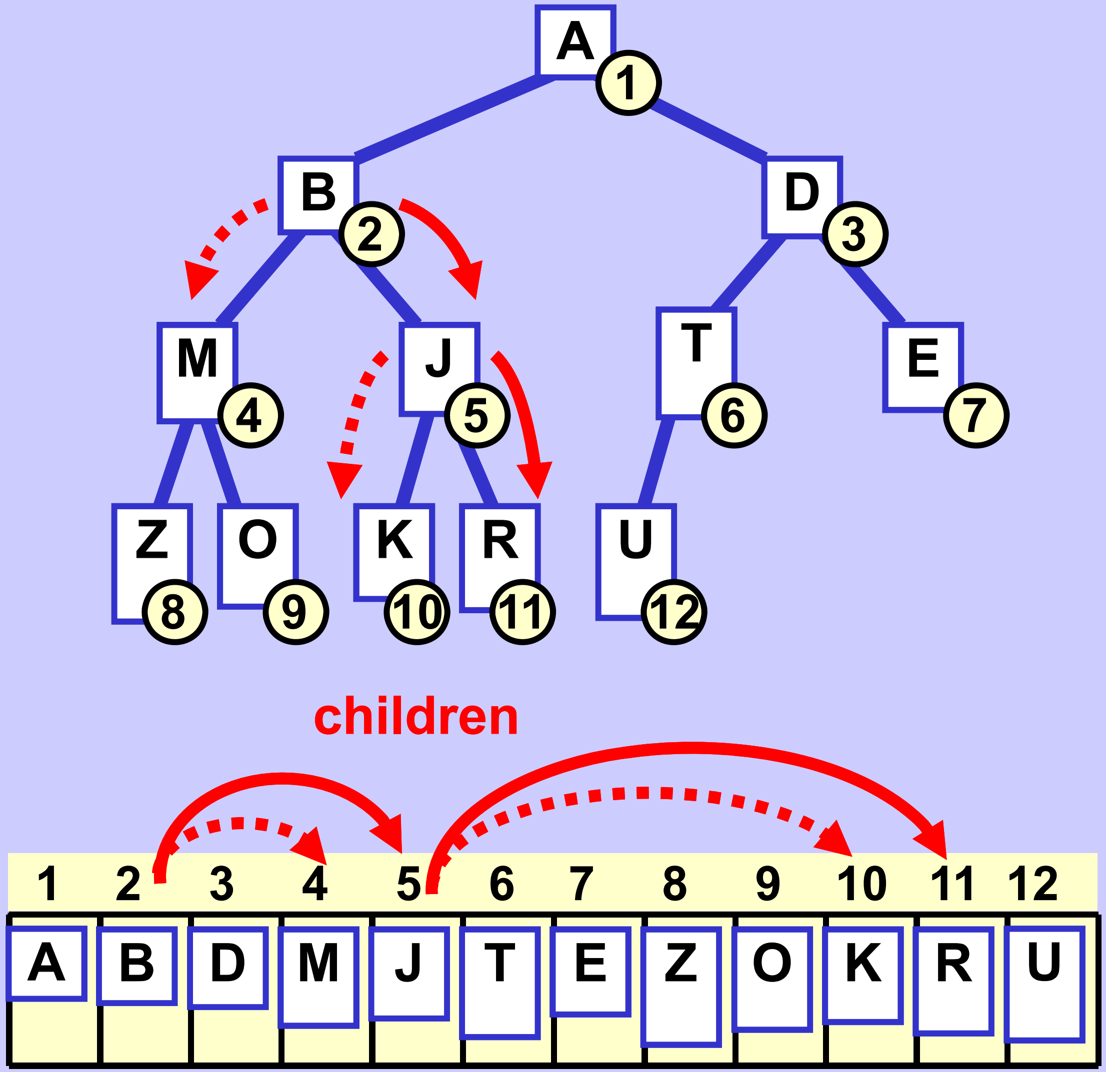
... make a heap in ⑥ ...

… and the whole heap is complete.

Heap in an array

Heap stored in an array



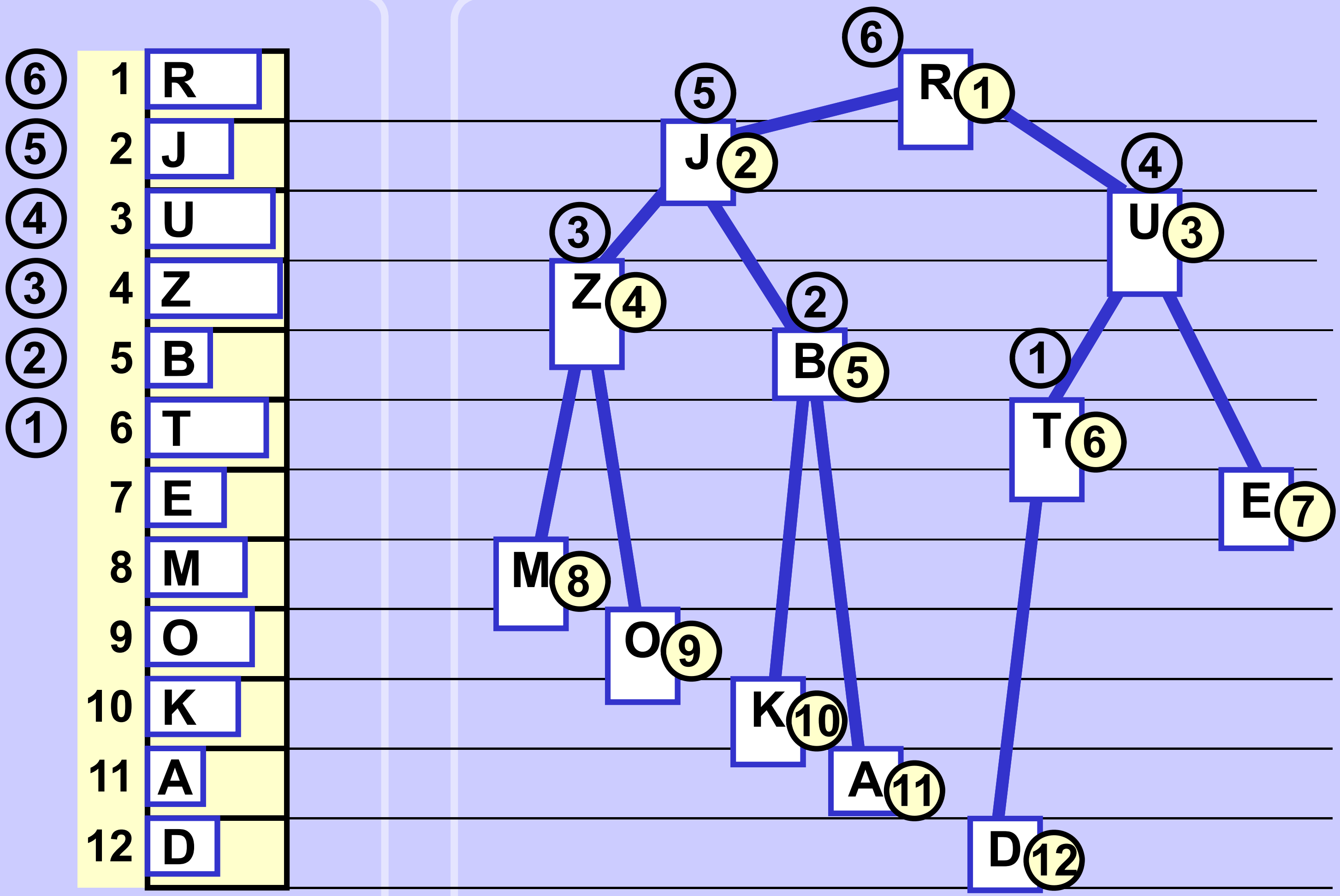


Heap in an array

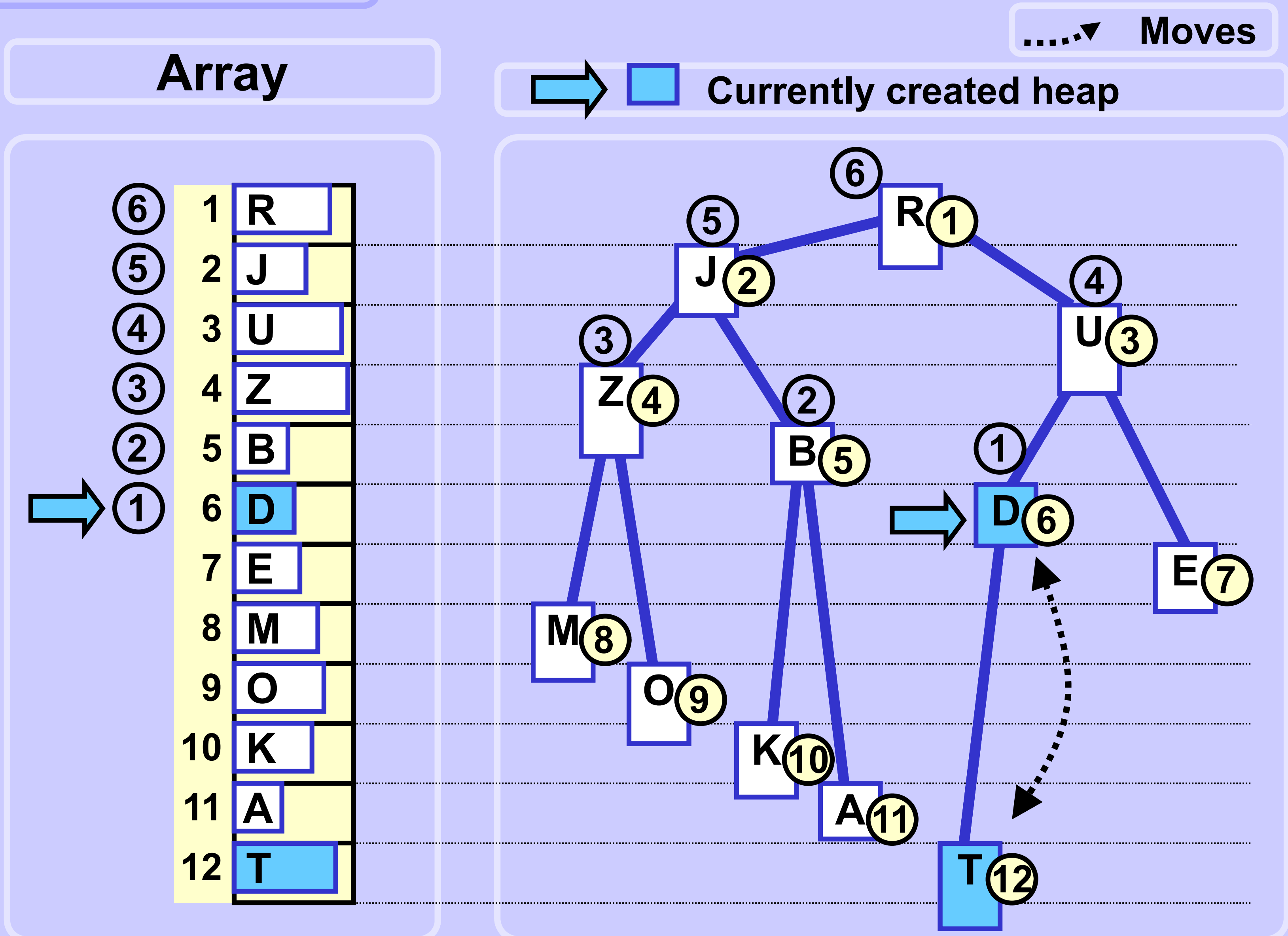
Array

Array elements ordered randomly

Not a heap



Heap in an array

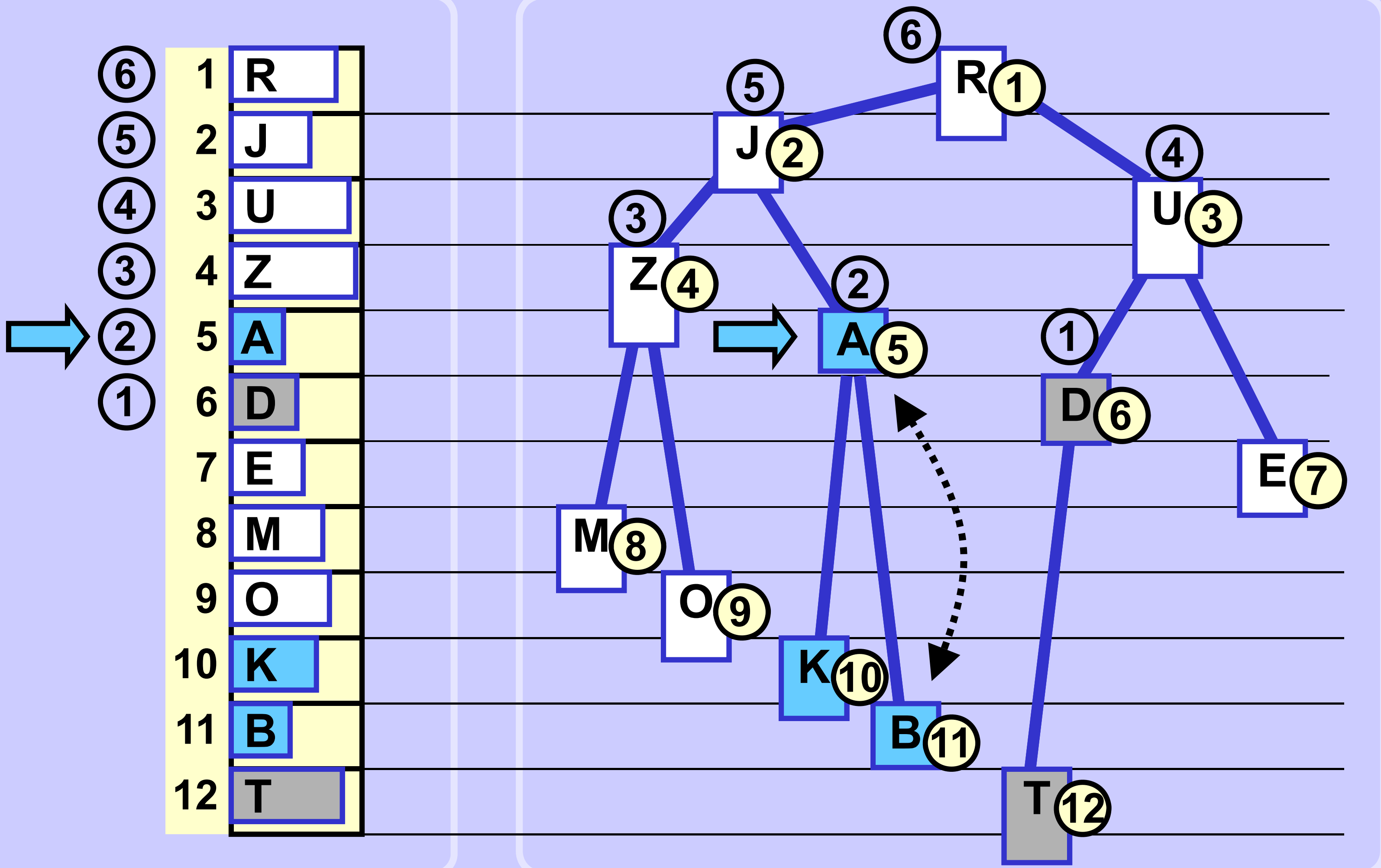


Heap in an array

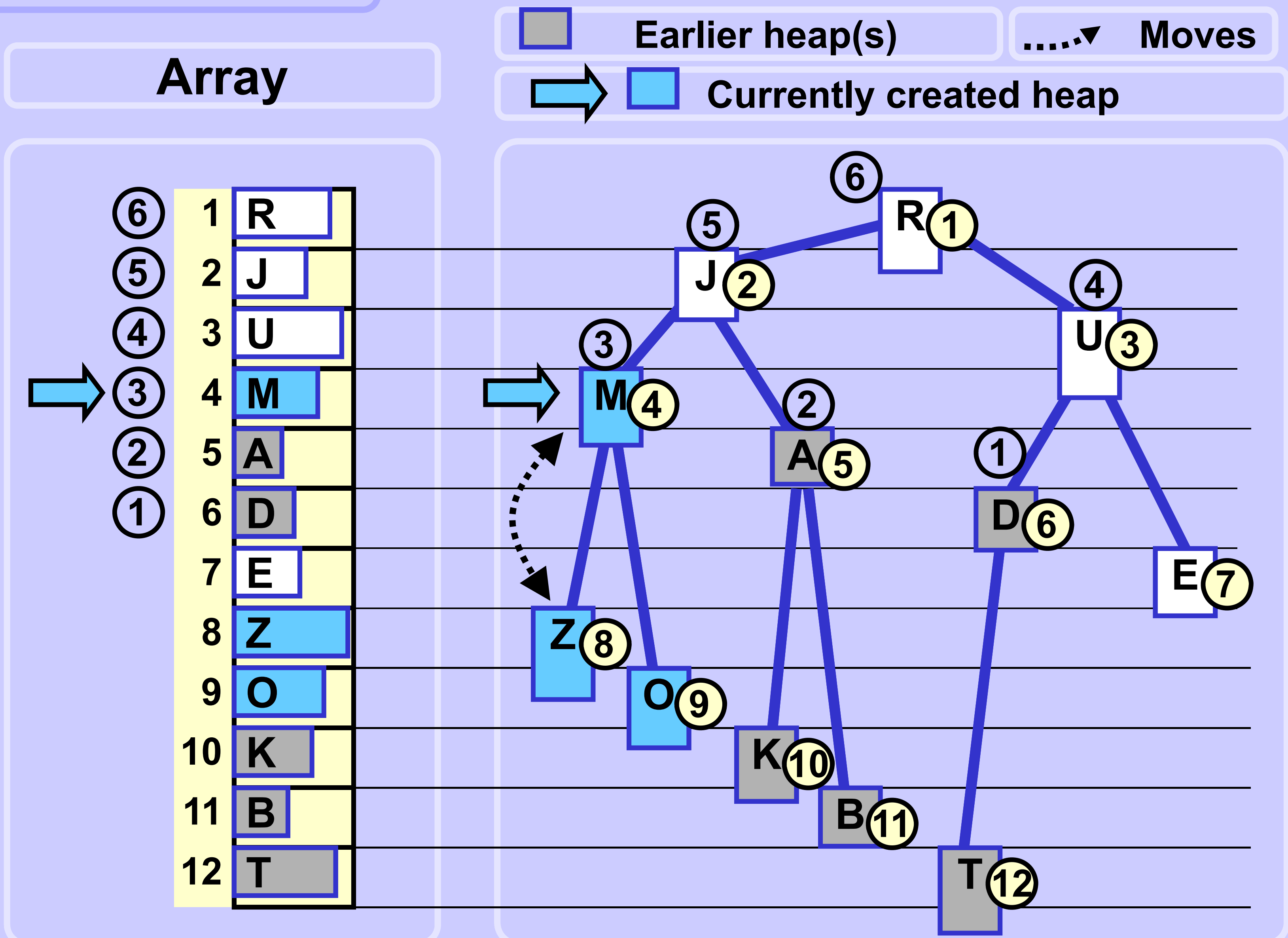
Earlier heap(s) Moves

Currently created heap

Array



Heap in an array



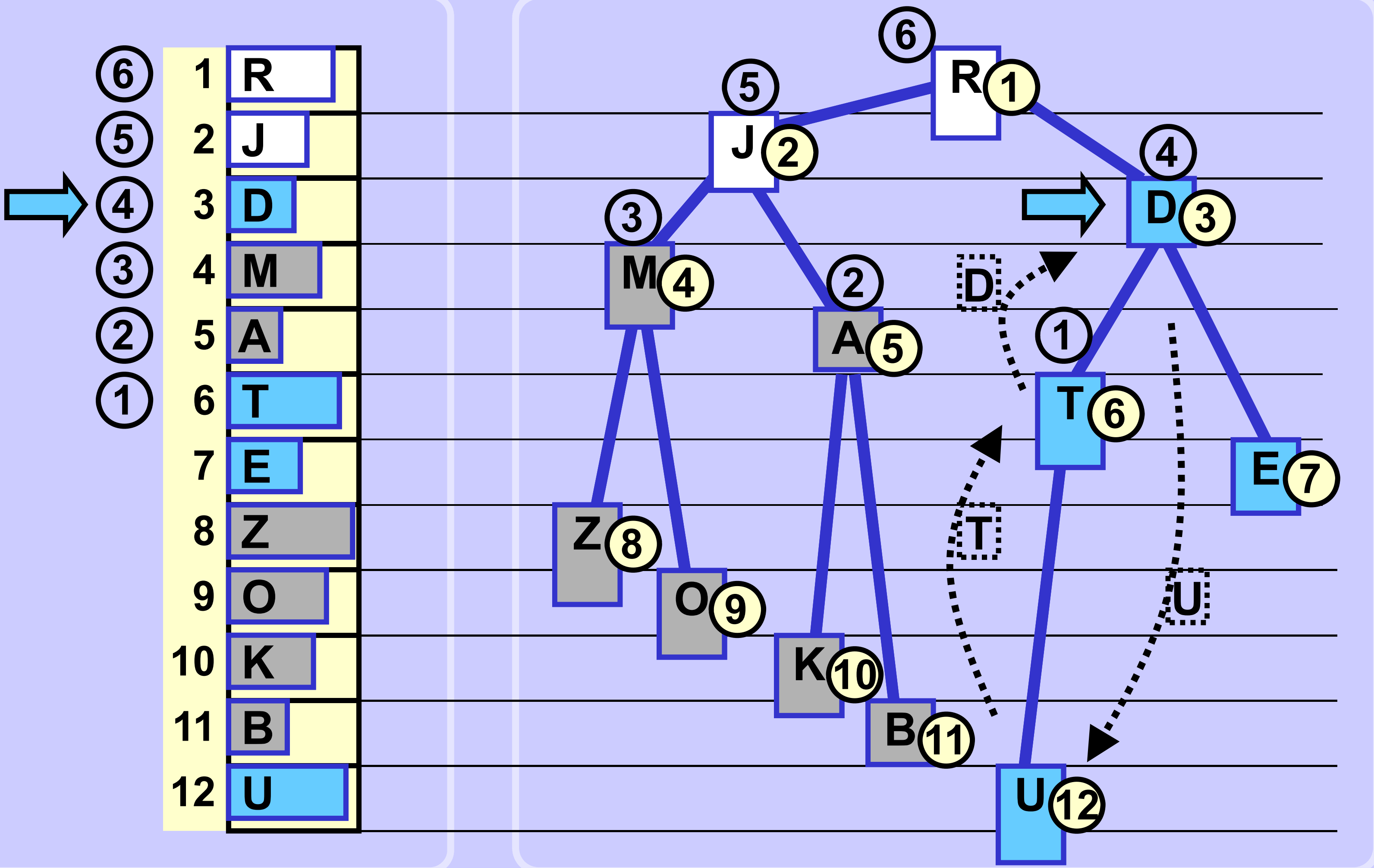
Heap in an array

Earlier heap(s)

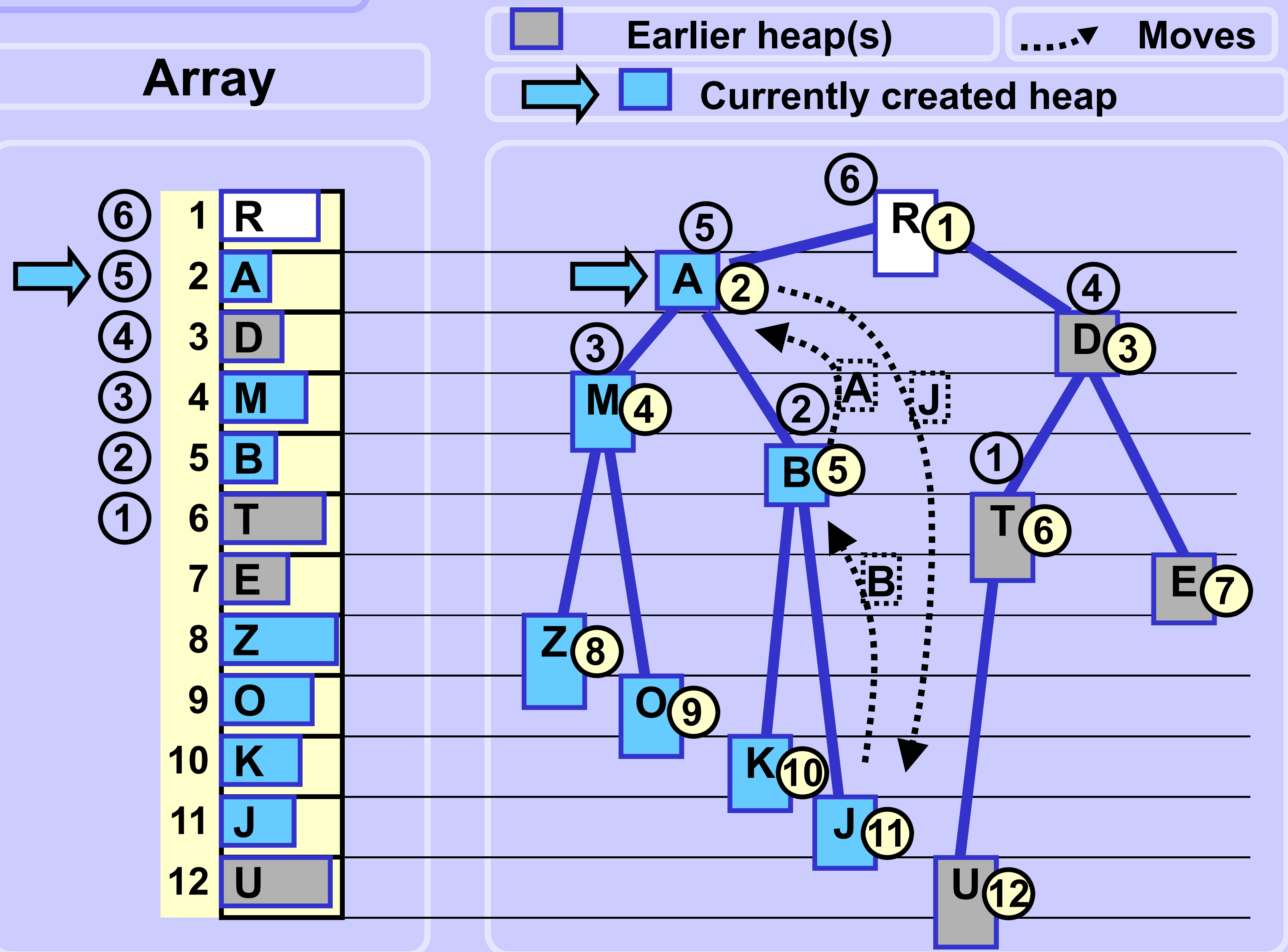
Moves

Currently created heap

Array



Heap in an array

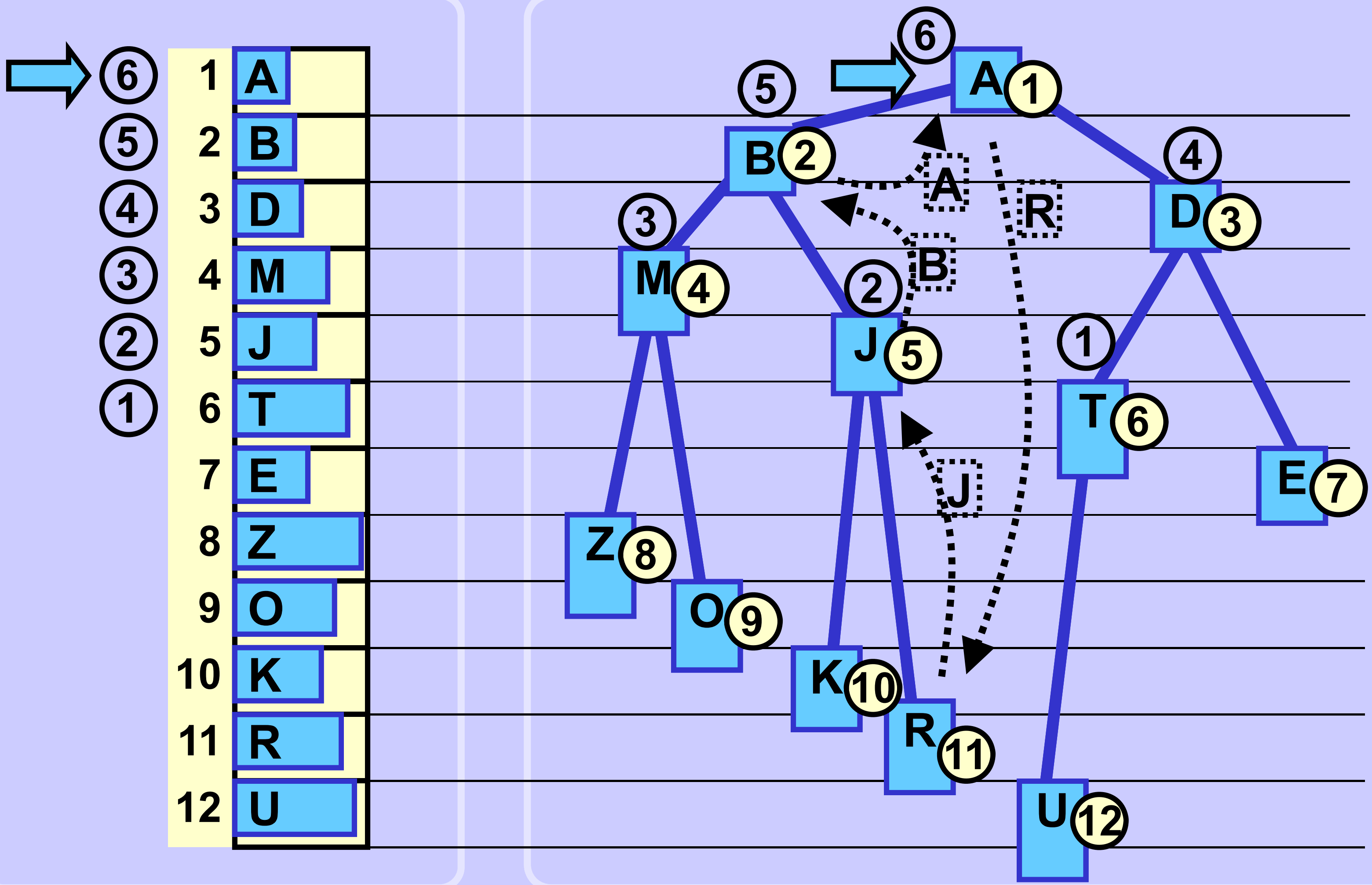


Heap in an array

Moves

Currently created heap

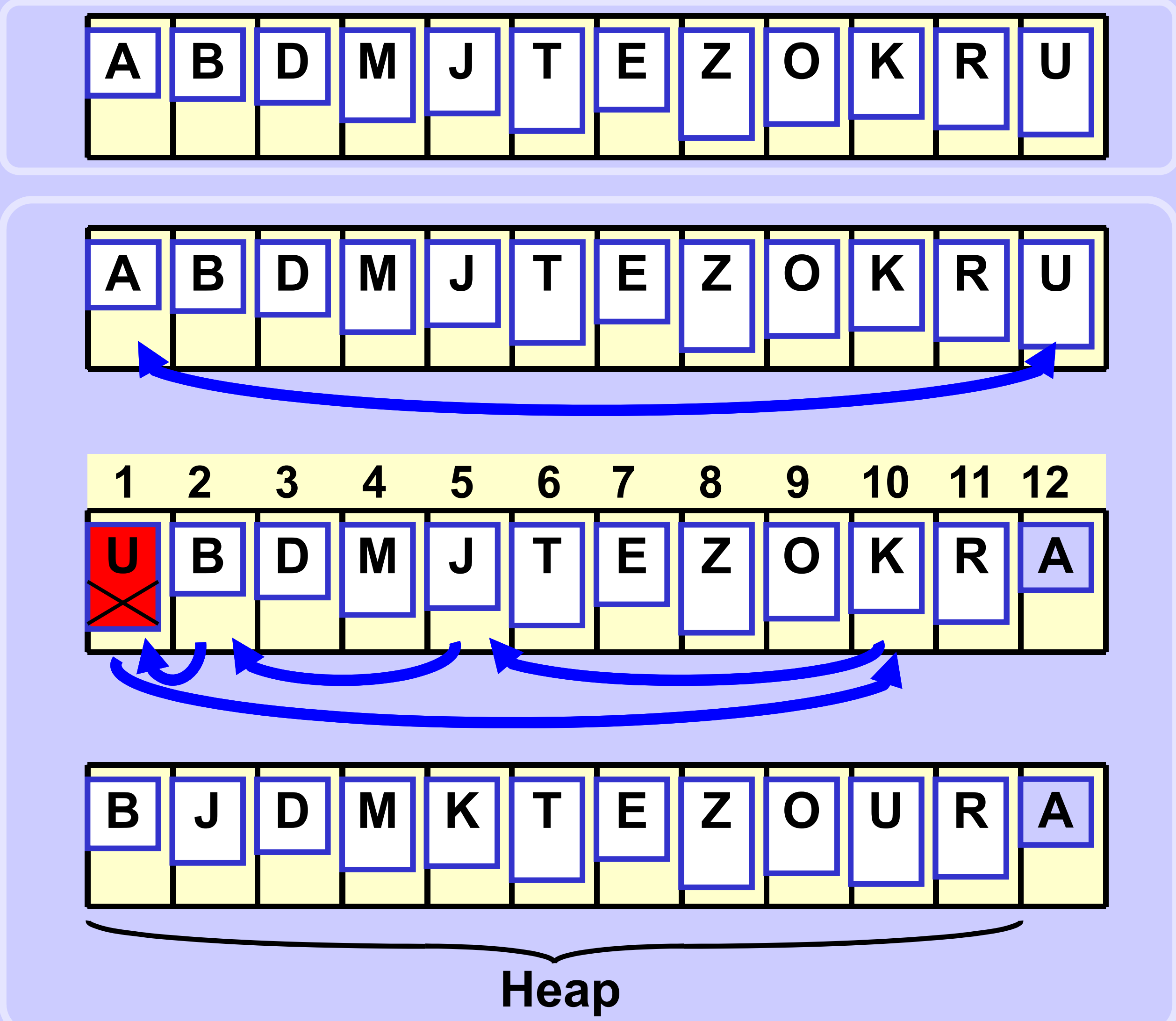
Array



Heap sort

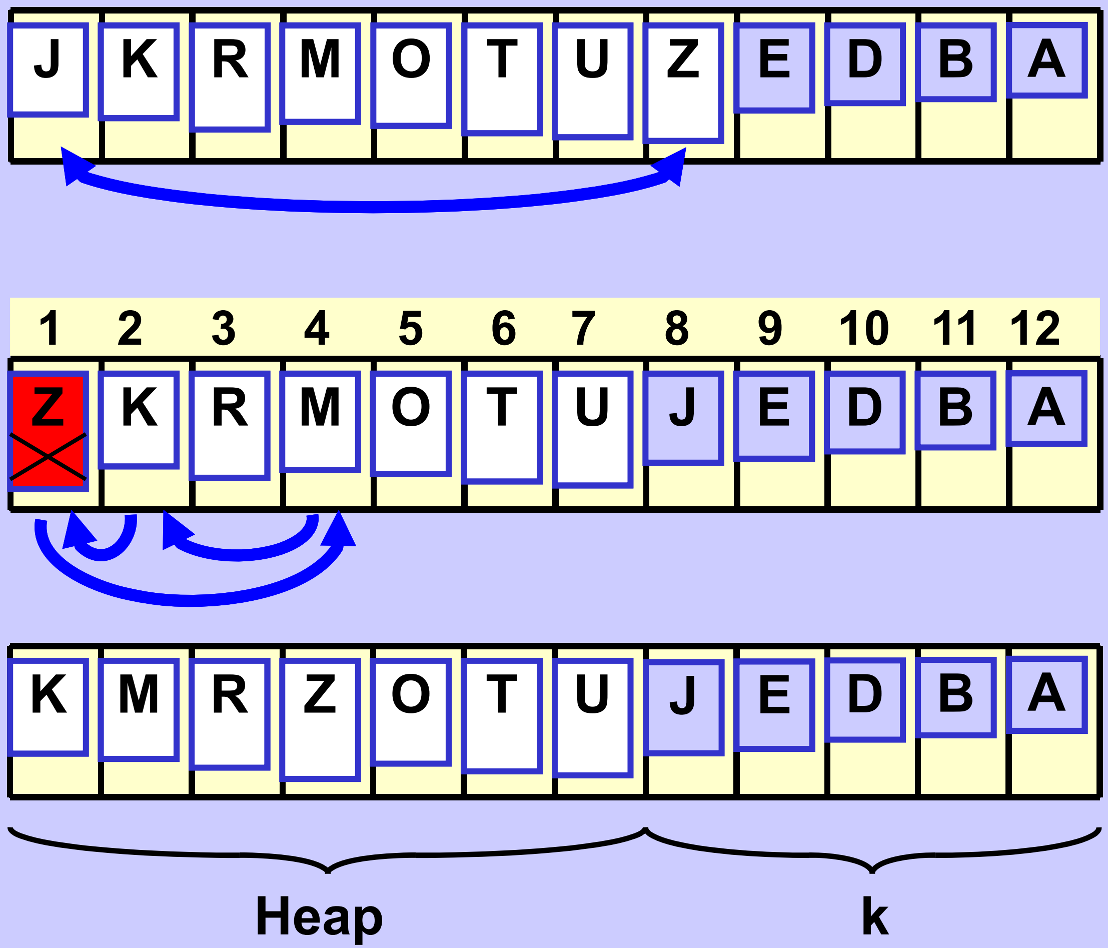
Heap

Step1 1



Heap sort

Step k



Heap sort

```
# beware!  array is arr[1] ... arr[n]

def heapSort (arr):
    n = len(arr)-1

    # create a heap
    for i in range(n//2, 0, -1): # progress backwards!
        repairTop(arr, i, n)

    for i in range(n, 1, -1):     # progress backwards!
        swap(arr, 1, i)
        repairTop(arr, 1, i-1)
```

Heap sort

```
def repairTop (arr, top, bottom):
    i = top     # arr[2*i] and arr[2*i+1]
    j = i*2     # are successors of arr[i]
    topVal = arr[top]

    # try to find a successor < topVal
    if j < bottom and arr[j] > arr[j+1]: j += 1

    # while successors < topVal move successors up
    while j <= bottom and topVal > arr[j]:
        arr[i] = arr[j]
        i = j; j = j*2     # move to next suceessor
        if j < bottom and arr[j] > arr[j+1]: j += 1

    # put topVal to its correct place
    arr[i] = topVal
```

Heap sort

repairTop operation worst case ... $\log_2(n)$ (n = heap size)

make a heap ... n/2 repairTop calls

$\log_2(n/2) + \log_2(n/2+1) + \ldots + \log_2(n) \leq (n/2)(\log_2(n)) = \underline{\underline{O(n \cdot \log_2(n))}}$

sort the heaps ... n-1 repairTop calls, worst case:

$\log_2(n) + \log_2(n-1) + \ldots + 1 \leq n \cdot \log_2(n) = O(n \cdot \log_2(n))$

surprisingly, also the best case = $\underline{\underline{\Theta(n \cdot \log_2(n))}}$

total ... make a heap + sort the heap = $\underline{\underline{\Theta(n \cdot \log_2(n))}}$

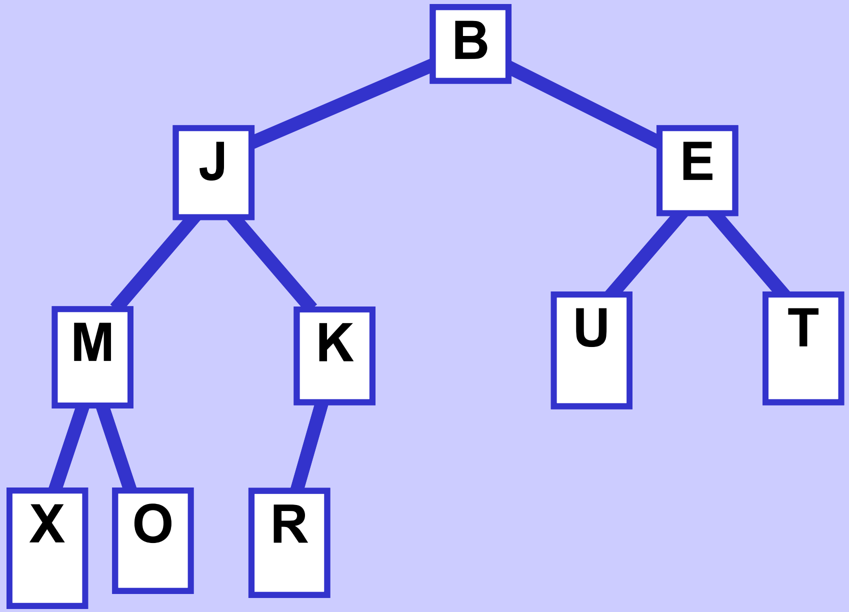
Asymptotic complexity of Heap sort is $\Theta(n \cdot \log_2(n))$.

Heap sort is not stable.

Priority queue

Operations
- Insert or Enqueue
- Front, Top, Peek -- read topmost element
- Dequeue, Pop, Poll -- delete topmost element.

The element with the smallest value (biggest value in max-heaps) of all elements in the heap is always at the top.



Priority queue might be implemented using a heap.

Officially:

" A *binary* heap".

Priority queue implemented with binary heap -- operations

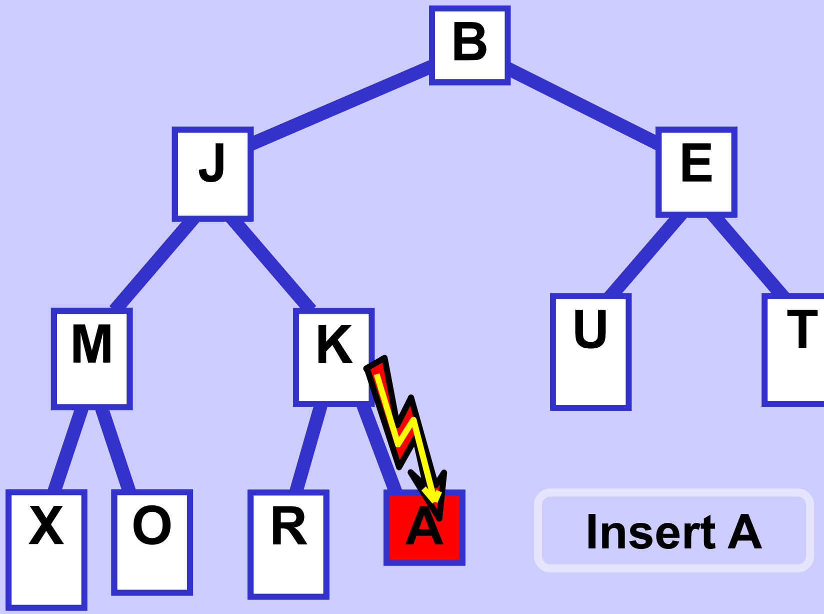
Read the topmost element (Front, Top, Peek, ...) .

Obvious.

Delete the topmost element (Dequeue, Pop, Poll, ...) =
Remove the top and repair the heap.

As before.

Insert an element to the queue (Insert, Enqueue, ...)



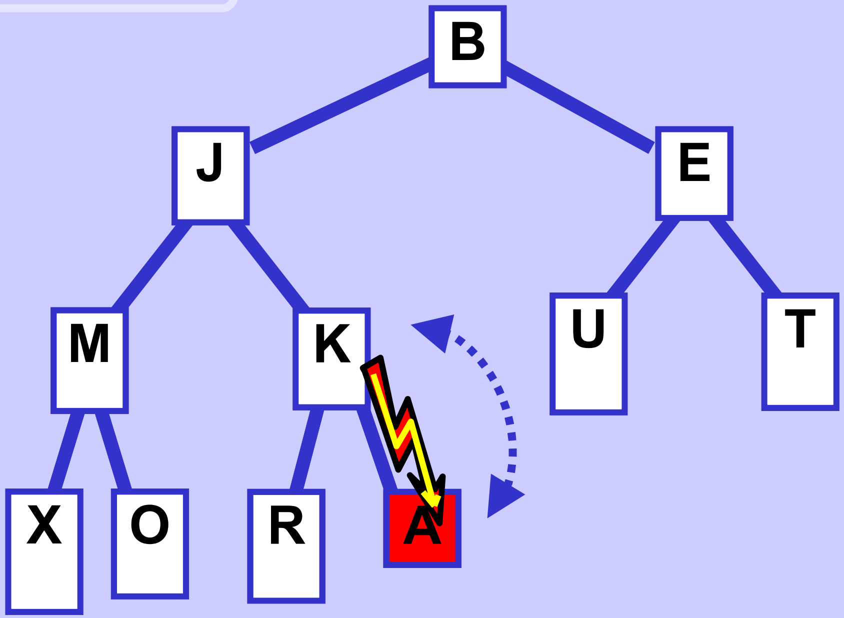
Insert the element
at the end of the queue
(end of the heap).

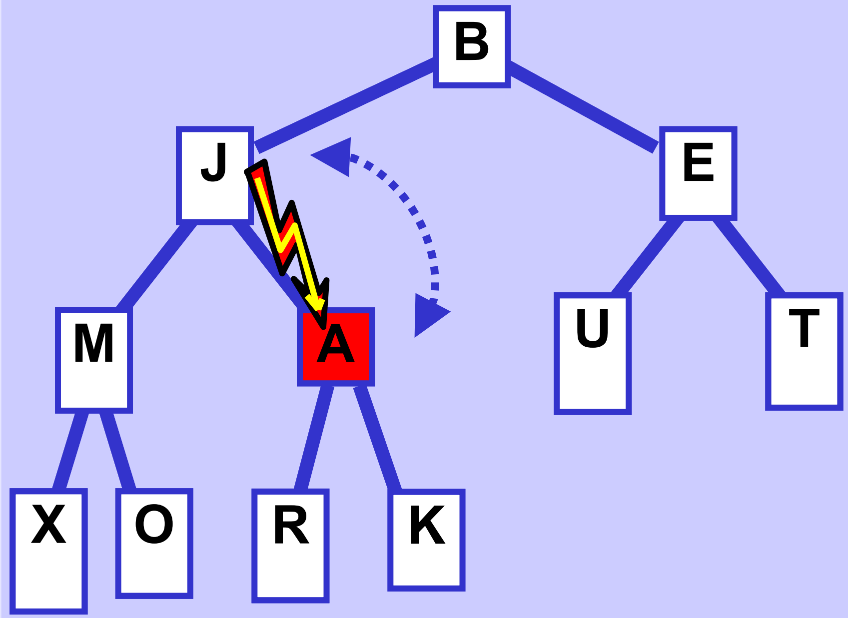
In most cases, this
violates the heap property

and the heap has to be
repaired.

Priority queue implemented with binary heap -- Insert

Insert A



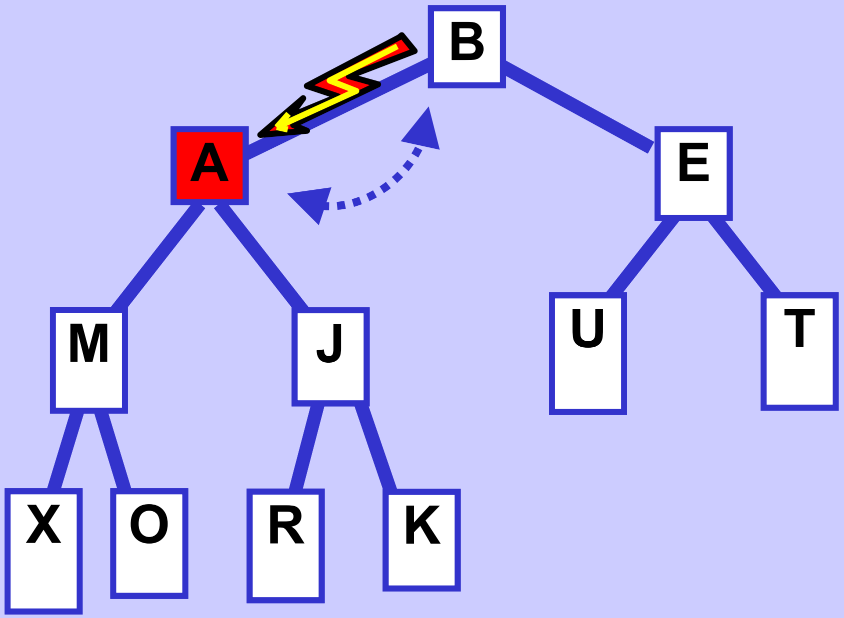


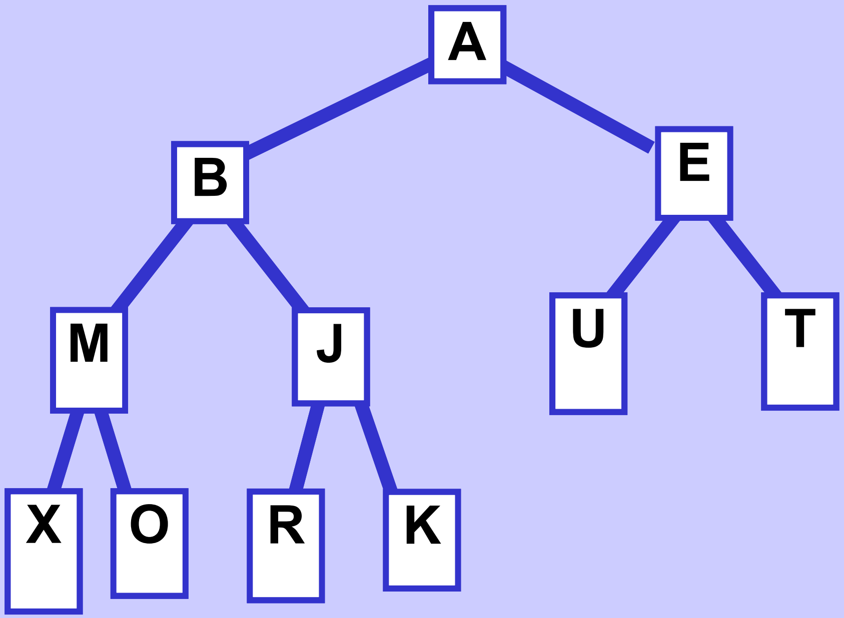
Heap property is violated,
swap the element with its parent.

Heap property is still violated,
swap the element with its parent.

Priority queue implemented with binary heap -- Insert

Inserting A



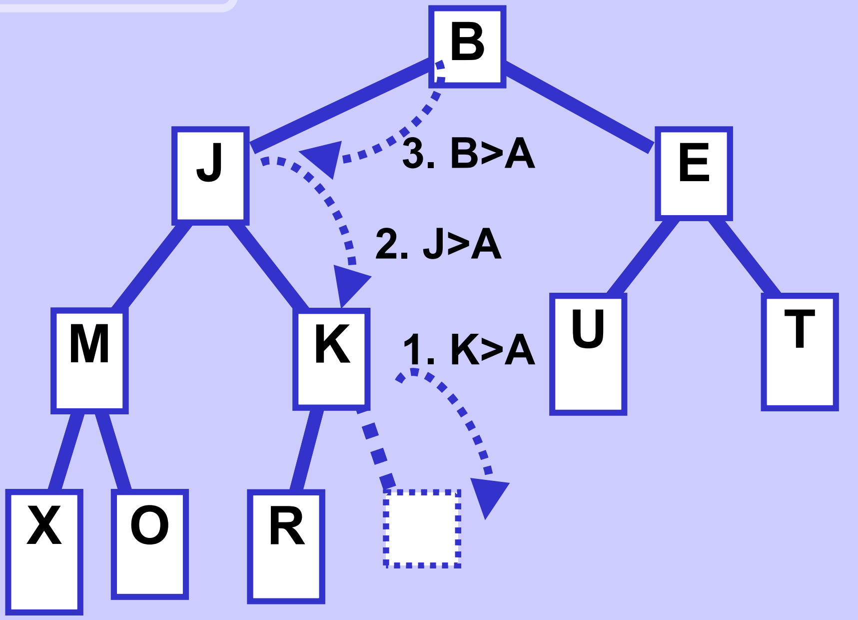


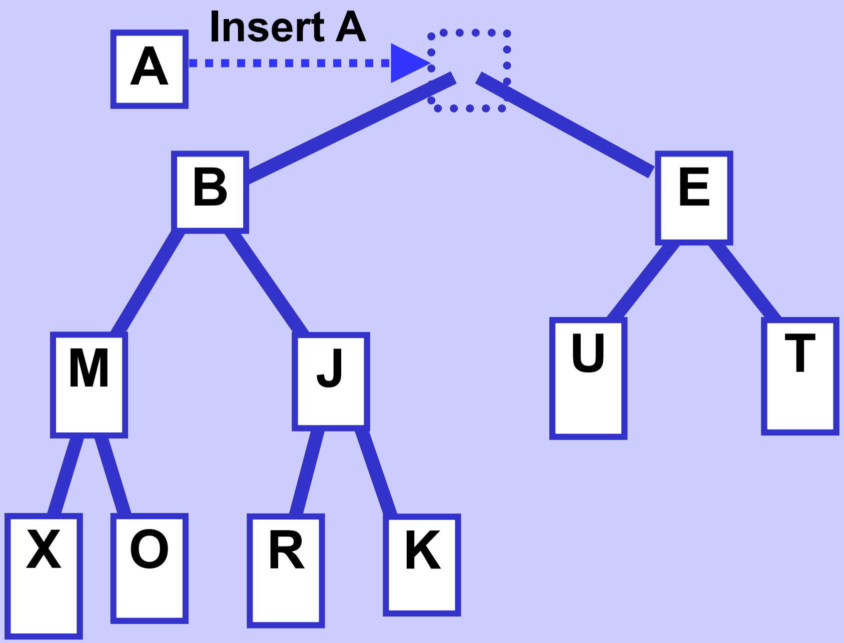
Heap property is still violated,
swap the element with its parent.

Heap property is respected,
the inserted element has found
its place in the queue (heap).

Binary heap -- Insert element more effectively

Insert A





Do not insert the element at the end of the queue.
First, find its place and while searching move down other elements encountered in the search.

Finally, store the inserted element at its correct position.

Binary heap – Insert

```
# beware!  array is arr[1] ... arr[n]
# bottom == ndx of last elem
def heapInsert(arr, x, bottom):
    bottom += 1   # expand the heap space
    j = bottom
    i = j/2       # parent index

    while i > 0 and arr[i] > x:
        arr[j] = arr[i]        # move elem down the heap
        j = i; i /= 2          # move indices up the heap

    arr[i] = x                 # put inserted elem to its place
    return bottom
```

Insert -- Complexity

Inserting represents a traversal in a binary tree from a leaf to the root in the worst case. Therefore, the Insert complexity is $O(\log_2(n))$.