# Multiagent Systems (BE4M36MAS)

## Solving Normal-Form Games

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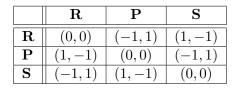
October 17, 2019

Previously ... on multi-agent systems.

- **1** Formal definition of a game  $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ 
  - $\mathcal{N}$  a set of players
  - *A* a set of actions
  - u outcome for each combination of actions
- 2 Pure strategies
- 3 Dominance of strategies
- 4 Nash equilibrium

... and now we continue ...

## **Rock Paper Scissors**



What is the best strategy to play in Rock-Paper-Scissors?

Every time we are about to play we randomly select an action we are going to use.

The concept of pure strategies is not sufficient.

### Definition (Mixed Strategies)

Let  $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$  be a normal-form game. Then the set of *mixed* strategies  $\mathcal{S}_i$  for player *i* is the set of all probability distributions over  $\mathcal{A}_i$ ;  $\mathcal{S}_i = \Delta(\mathcal{A}_i)$ .

Player selects a pure strategy according to the probability distribution.

We extend the utility function to correspond to expected utility:

$$u_i(s) = \sum_{a \in A} u_i(a) \prod_{j \in \mathcal{N}} s_j(a_j)$$

We can extend existing concepts (dominance, best response, ...) to mixed strategies.

## Dominance

#### Definition (Strong Dominance)

Let  $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$  be a normal-form game. We say that  $s_i$ strongly dominates  $s'_i$  if  $\forall s_{-i} \in \mathcal{S}_{-i}, u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$ .

#### Definition (Weak Dominance)

Let  $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$  be a normal-form game. We say that  $s_i$  weakly dominates  $s'_i$  if  $\forall s_{-i} \in \mathcal{S}_{-i}, u_i(s_i, s_{-i}) \ge u_i(s'_i, s_{-i})$  and  $\exists s_{-i} \in \mathcal{S}_{-i}$  such that  $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$ .

#### Definition (Very Weak Dominance)

Let  $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$  be a normal-form game. We say that  $s_i$  very weakly dominates  $s'_i$  if  $\forall s_{-i} \in \mathcal{S}_{-i}, u_i(s_i, s_{-i}) \ge u_i(s'_i, s_{-i})$ .

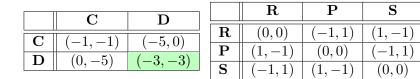
### Definition (Best Response)

Let  $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$  be a normal-form game and let  $BR_i(s_{-i}) \subseteq \mathcal{S}_i$ such that  $s_i^* \in BR_i(s_{-i})$  iff  $\forall s_i \in \mathcal{S}_i, u_i(s_i^*, s_{-i}) \ge u_i(s_i, s_{-i}).$ 

#### Definition (Nash Equilibrium)

Let  $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$  be a normal-form game. Strategy profile  $s = \langle s_1, \dots, s_n \rangle$  is a Nash equilibrium iff  $\forall i \in \mathcal{N}, s_i \in BR_i(s_{-i})$ .

# Existence of Nash equilibria?



## Theorem (Nash)

Every game with a finite number of players and action profiles has at least one Nash equilibrium in mixed strategies.

## Definition (Support)

The *support* of a mixed strategy  $s_i$  for a player i is the set of pure strategies  $\text{Supp}(s_i) = \{a_i | s_i(a_i) > 0\}.$ 

#### Question

Assume Nash equilibrium  $(s_i, s_{-i})$  and let  $a_i \in \text{Supp}(s_i)$  be an (arbitrary) pure strategy from the support of  $s_i$ . Which of the following possibilities can hold?

$$u_i(a_i, s_{-i}) < u_i(s_i, s_{-i})$$

$$u_i(a_i, s_{-i}) = u_i(s_i, s_{-i})$$

 $u_i(a_i, s_{-i}) > u_i(s_i, s_{-i})$ 

# Support of Nash Equilibria

#### Corollary

Let  $s \in S$  be a Nash equilibrium and  $a_i, a'_i \in A_i$  are actions from the support of  $s_i$ . Now,  $u_i(a_i, s_{-i}) = u_i(a'_i, s_{-i})$ .

Can we exploit this fact to find a Nash equilibrium?

# Finding Nash Equilibria

		R
U	(2,1)	(0, 0)
D	(0,0)	(1,2)

Column player (player 2) plays L with probability p and R with probability (1-p). In NE it holds

$$\mathbb{E}u_1(\mathbf{U}) = \mathbb{E}u_1(\mathbf{D})$$
$$2p + 0(1-p) = 0p + 1(1-p)$$
$$p = \frac{1}{3}$$

Similarly, we can compute the strategy for player 1 arriving at  $(\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})$  as Nash equilibrium.

Can we use the same approach here?

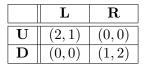
	L	С	R
U	(2,1)	(0, 0)	(0, 0)
M	(0,0)	(1,2)	(0, 0)
D	(0, 0)	(0, 0)	(-1, -1)

Not really... No strategy  $s_i$  of the row player ensures  $u_{-i}(s_i,L)=u_{-i}(s_i,C)=u_{-i}(s_i,R)$  :-(

#### Can something help us?

Iterated removal of dominated strategies. Search for a possible support (enumeration of all possibilities).

## Maxmin



Recall that there are multiple Nash equilibria in this game. Which one should a player play? This is a known equilibrium-selection problem.

Playing a Nash strategy does not give any guarantees for the expected payoff. If we want guarantees, we can use a different concept – maxmin strategies.

## Definition (Maxmin)

The maxmin strategy for player i is  $\arg \max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$ and the maxmin value for player i is  $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$ .

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#### Definition (Minmax, two-player)

In a two-player game, the minmax strategy for player i against player -i is  $\arg\min_{s_i}\max_{s_{-i}}u_{-i}(s_i, s_{-i})$  and the minmax value for player -i is  $\min_{s_i}\max_{s_{-i}}u_{-i}(s_i, s_{-i})$ .

Maxmin strategies are conservative strategies against a worst-case opponent.

Minmax strategies represent punishment strategies for player -i.

### What about zero-sum case? How do

 $\blacksquare$  player i 's maxmin,  $\max_{s_i} \min_{s_{-i}} u_i(s_i,s_{-i})$  , and

■ player *i*'s minmax,  $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$  relate?

$$\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = -\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$$

## ... but we can prove something stronger ...

## Theorem (Minimax Theorem (von Neumann, 1928))

In any finite, two-player zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and the minmax value of his opponent.



Consequences:

- $1 \max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$
- 2 we can safely play Nash strategies in zero-sum games
- 3 all Nash equilibria have the have the same payoff (by convention, the maxmin value for player 1 is called *value of the game*).

We can now compute Nash equilibrium for two-player, zero-sum games using a linear programming:

s.t. 
$$\sum_{a_1 \in \mathcal{A}_1} s(a_1)u_1(a_1, a_2) \ge U \qquad (1)$$
$$\sum_{a_1 \in \mathcal{A}_1} s(a_1)u_1(a_1, a_2) \ge U \qquad \forall a_2 \in \mathcal{A}_2 \qquad (2)$$
$$\sum_{a_1 \in \mathcal{A}_1} s(a_1) = 1 \qquad (3)$$
$$s(a_1) \ge 0 \qquad \forall a_1 \in \mathcal{A}_1 \qquad (4)$$

Computing a Nash equilibrium in zero-sum normal-form games can be done in polynomial time.