Robust statistics

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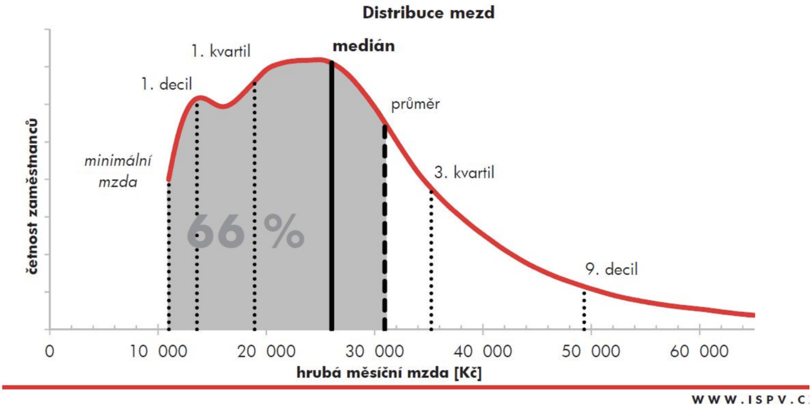
Motivation

Is an average salary a good measure?

Motivation

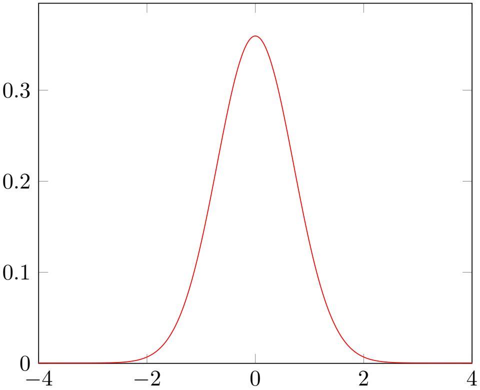

Distribuce mezd
medián
1. kvartil
1. decil
průměr
minimální
mzda
3. kvartil
66 %
9. decil
četnost zaměstnanců
0
10 000
20 000
30 000
40 000
50 000
60 000
hrubá měsíční mzda [Kč]
WWW.ISPV.C

Goals of robust statistics

It should not be affected by

- the presence of outliers
- or in-correctness of assumed probability distribution.

Which distribution is this?



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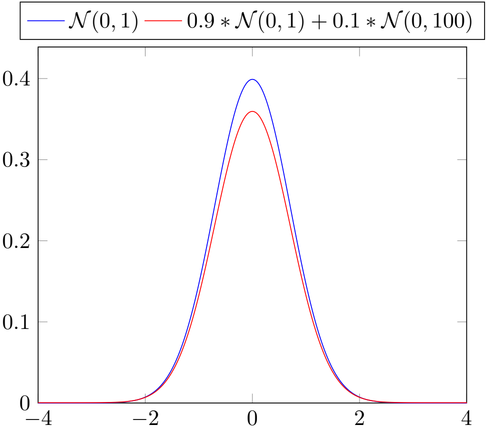
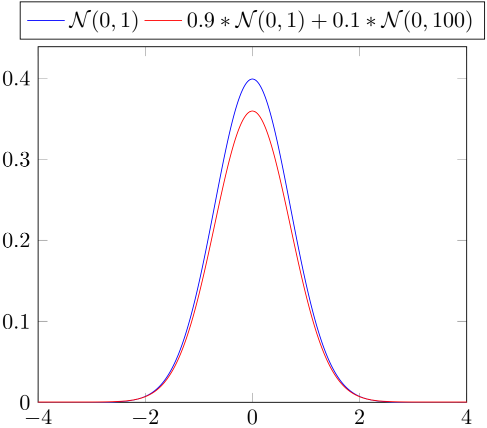
Motivation



$\mathcal{N}(0,1)$
$0.9 * \mathcal{N}(0,1) + 0.1 * \mathcal{N}(0,100)$
0.4
0.3
0.2
0.1
0
−4
−2
0
2
4

;

Motivation



;

mean estimated from 1000 samples: $-5 \cdot 10^{-3}$, 0.49

median estimated from 1000 samples: -0.012, -0.013

Plan

Breakdown Point

Breakdown Point: the largest proportion of sample observations which may be given arbitrary values without taking the estimator to a limit uninformative about the parameter being estimated.

Example: Breakdown point

Breakdown point of

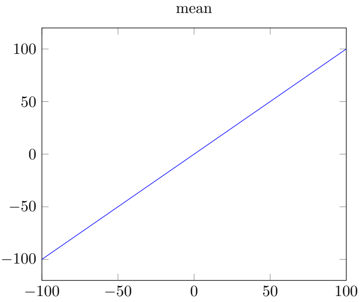
- mean is 0,
- median is 50%.

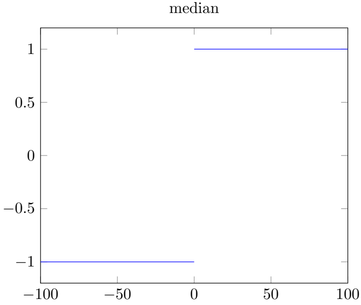
Influence function

$$\mathrm{IF}(x|p,\eta) = \lim_{\varepsilon \to 0} \frac{\eta((1-\varepsilon)p + \varepsilon\delta_x) - \eta(p)}{\varepsilon}$$

p — probability distribution
η — estimator
δ — polluting probability distribution function (dirac)

Influence function of mean and max





$$\mathrm{IF}(x|p,\eta) = \lim_{\varepsilon\to 0} \frac{\eta((1-\varepsilon)p+\varepsilon\delta_x)-\eta(p)}{\varepsilon}$$

Gross Error Sensitivity

$$\mathrm{IF}(x|p,\eta) = \lim_{\varepsilon \to 0} \frac{\eta((1-\varepsilon)p + \varepsilon\delta_x) - \eta(p)}{\varepsilon}$$

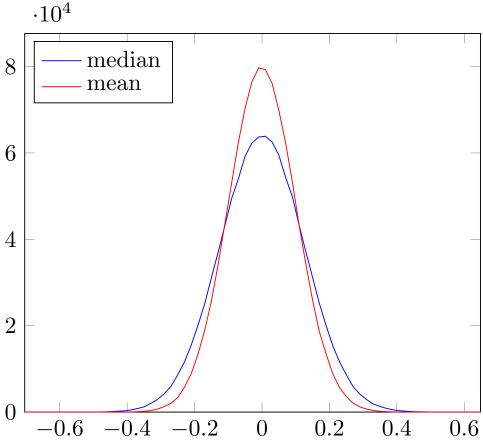
$$\mathrm{GES}(p,\eta) = \sup_x |\mathrm{IF}(x)|$$

p — probability distribution
η — estimator
δ — polluting probability distribution function (dirac)

How to measure efficiency

How is the sampling distribution of the estimator spread about the true value?

How to measure efficiency



Distribution of mean and median estimates of *true mean* from 100 samples from $\mathcal{N}(0,1)$.

Example of comparison of efficiency:

Assuming x_i are i.i.d samples from p

- $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} x_i \sim \mathcal{N}\left(\mu, \frac{\sigma_p^2}{n}\right)$
- $\mathrm{Med}(X_n) = \mathrm{med}\{x_1, \ldots, x_n\} \sim \mathcal{N}\left(\mu, \frac{1}{4p^2(\mu)n}\right)$

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- Median and mean estimates are equally precise, iff

$$n_1 = \frac{4p^2(\mu)n_2}{\sigma_p^2}.$$

Asymptotic relative efficiency

Asymptotic relative efficiency (ARE) is defined as

$$\mathrm{ARE}(\hat{\eta}_1, \hat{\eta}_2, p) = \frac{V_2}{V_1},$$

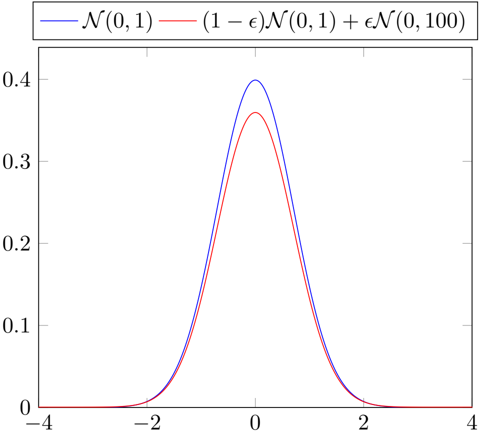
where $\frac{V_1}{n}$, $\frac{V_2}{n}$ are variances of estimators $\hat{\eta}_1$, $\hat{\eta}_2$ of a parameter μ of probability distribution p.

Example: mean and median of Gaussian distribution

- sample mean $\bar{X}_n \sim (\mu, \frac{\sigma^2}{n})$
- sample media $\mathrm{Med}(X_n) \sim \mathscr{N}\left(\mu, \frac{1}{4p^2(\mu)n}\right)$
- Asymptotic relative efficiency of median to mean is

$$\mathrm{ARE}(\mathrm{Med}, \bar{X}, \mathscr{N}) = 4p^2(\mu)\sigma^2$$

Example: mean and median of Gaussian mixture



For $\varepsilon > 0.1 \Rightarrow \mathrm{ARE}(\mathrm{Med}, \bar{X}, \mathcal{N}) > 1$

Plan

Estimators of location

- mean
- median
- q%-trimmed
- q%-winsorized
- Hodges-Lehmann

Mean

$$\{-39.61, -26.29, -1.07, -0.92, -0.85, -0.16, 0.93, 1.91, 2.18, 133.65\}$$

- mean $\frac{1}{n}\sum_i x_i = 6.97$
- Zero breakdown
- Optimal if samples follows Normal distribution.

Median

{-39.61, -26.29, -1.07, -0.92, -0.85, -0.16, 0.93, 1.91, 2.18, 133.65}

- median $\text{median}\{x_1, \ldots, x_{10}\} = -0.51$
- 50% breakdown
- ARE = 0.637 for Normal distribution

q%-trimmed

$$\{-39.61, -26.29, -1.07, -0.92, -0.85, -0.16, 0.93, 1.91, 2.18, 133.65\}$$

- calculate mean from samples $\left\{x | x_{q\%} \leq x \leq x_{1-q\%}\right\}$
- mean $\frac{1}{|\mathscr{X}_q|} \sum_{x \in \mathscr{X}_q} x_i = -0.41$
- q% breakdown
- ARE $= 0.943$ for Normal distribution with $q = 10\%$

q%-Windsorized

$\{-1.07, -1.07, -1.07, -0.92, -0.85, -0.16, 0.93, 1.91, 1.99, 1.99\}$

- replace samples outside $\langle x_{q\%}, x_{1-q\%} \rangle$ by bounds, return mean.
- mean $\frac{1}{N} \sum_{X \in \tilde{\mathscr{X}}_q} x_i = 0.33$
- q% breakdown

Hodges-Lehman

$$\{-39.61, -26.29, -1.07, -0.92, -0.85, -0.16, 0.93, 1.91, 2.18, 133.65\}$$

- $\mathrm{HL} = \mathrm{med}\left\{\frac{x_i + x_j}{2} | i, j \in N\right\} = -0.03$
- 0.29 breakdown
- ARE = 0.955 for Normal distribution

Plan

Question

Why mean is so popular?

ML estimate of location of Normal distribution

Assuming $x \sim \mathcal{N}(\mu, \sigma^2)$, then *maximum likelihood estimate* (ML) of μ from $\{x_1, \ldots, x_n\}$ is

$$\arg\max_{\mu} \mathscr{L} = \prod_i \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x_i - \mu)^2}$$

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$$\arg\max_{\mu} \log L = -\frac{1}{2\sigma^2} \sum (x_i - \mu)^2 - \log \sqrt{2\pi}\sigma$$

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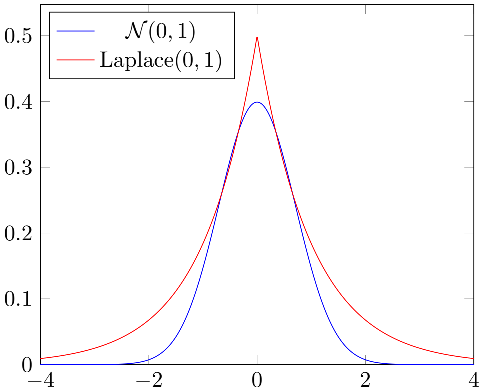
$$\mu = \frac{1}{n} \sum_i x_i$$

ML estimate of location of Laplace distribution



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$$0 = \sum_i \mathrm{sgn}(x_i - \mu)$$

Question

Can we generalize this?

ML estimate of location of exponential class

Assuming $x \sim \frac{1}{Z} e^{-\rho(\frac{x-\mu}{\sigma})}$, then *maximum likelihood estimate* (ML) of μ from $\{x_1, \ldots, x_n\}$ is

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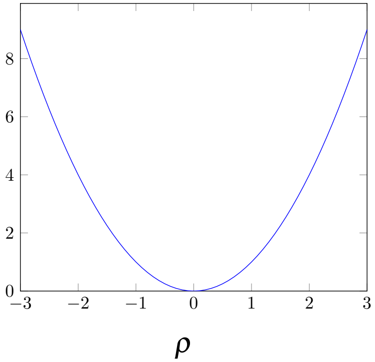
$$\arg\max_{\mu} \log L = -\sum_i \rho(\frac{x_i - \mu}{\sigma}) - \log Z$$

$$0 = \sum_i \rho'(\frac{x_i - \mu}{\sigma}).$$

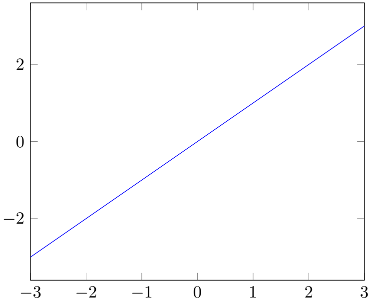
Normal distribution

- $\rho = \frac{1}{2}x^2$



Normal distribution

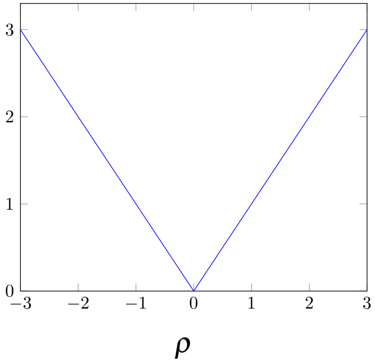
- $\rho = \frac{1}{2}x^2$
- $\rho' = x$
- $\mu = \frac{1}{n}\sum_i x_i$



ρ'

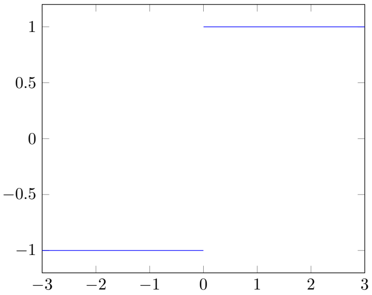
Laplace distribution

- $\rho = \frac{1}{2}|x|$



Laplace distribution

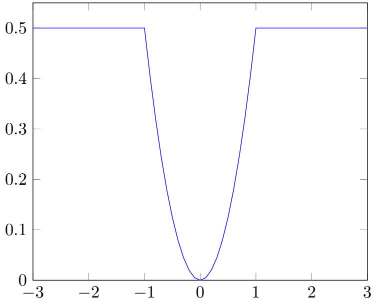
- $\rho = \frac{1}{2}|x|$
- $\rho' = \mathrm{sgn}(x)$
- $0 = \sum_i \mathrm{sgn}(x_i - \mu)$



ρ'

Huber loss (not called Huber)

- $\rho = \begin{cases} \frac{x^2}{2} & |x| < a \\ \frac{a^2}{2} & |x| \geq a \end{cases}$



ρ

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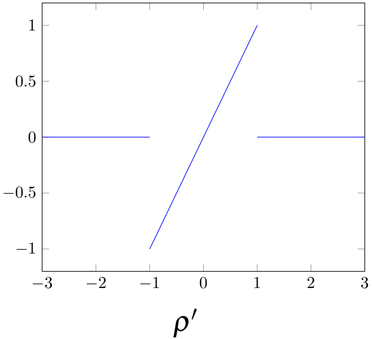
- $\rho = \begin{cases} \frac{x^2}{2} & |x| < a \\ \frac{a^2}{2} & |x| \geq a \end{cases}$
- $\rho' = \begin{cases} x & |x| < a \\ 0 & |x| \geq a \end{cases}$
- $\mu = \frac{1}{n_{<a}} \sum_{i|abs(x_i)<a} x_i$
- trimming



Huber loss (called Huber)

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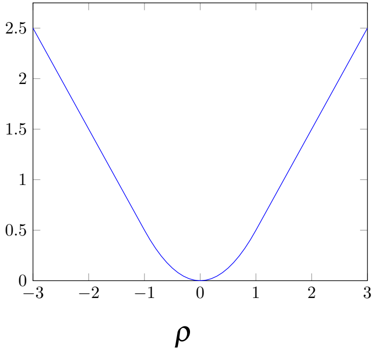

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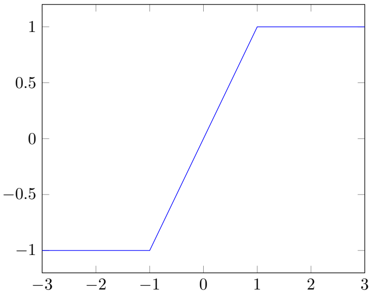
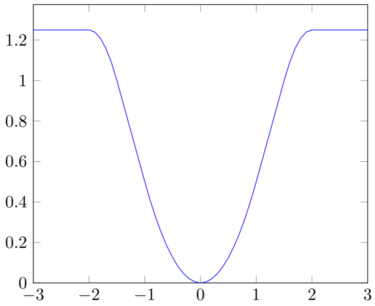
- $\rho' = \begin{cases} x & |x| < a \\ a \cdot \mathrm{sgn}(x) & |x| \geq a \end{cases}$
- $\mu = \frac{1}{n}\left[\sum_{i|abs(x_i)<a} x_i + n_{>a} \cdot a\right]$
- Windsorizing



ρ'

Hampel loss

$$\rho = \begin{cases} \frac{x^2}{2} & 0 \leq x < a \\ ax - \frac{a^2}{2} & a \leq x < b \\ \frac{a(x-c)^2}{2(b-c)} + \frac{1}{2}a(b+c-a) & b \leq x < c \\ \frac{1}{2}a(b+c-a) & c \leq x \end{cases}$$



ρ

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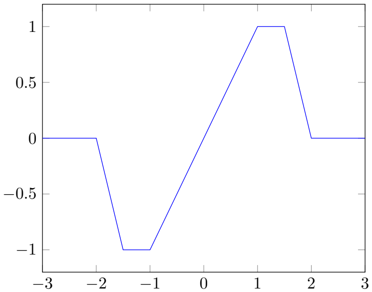

ρ'

Caveats of robust losses

- To use them you need to select parameters (scale or others)
- Robust losses might have unfavourable efficiency.
- Hampel loss is not convex — difficult to optimize.

Plan

Least-square regression is an M-estimator

Generative model behind OLS is

$$y = x^{\mathrm{T}}\beta + \varepsilon, \varepsilon \sim \mathcal{N}(0, \sigma^2).$$

$x, y \in \mathbb{R}^d$

Least-square regression is an M-estimator

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$$y = x^{\mathrm{T}}\beta + \varepsilon, \varepsilon \sim \mathcal{N}(0, \sigma^2).$$

Therefore

$$p(y|x, \beta, \sigma^2) \sim \mathcal{N}(x^{\mathrm{T}}\beta, \sigma^2)$$

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Therefore

$$p(y|x, \beta, \sigma^2) \sim \mathcal{N}(x^{\mathrm{T}}\beta, \sigma^2)$$

and

$$\hat{\beta} = \arg\min_{\beta} \sum_i (x_i^{\mathrm{T}}\beta - y_i)^2.$$

$x, y \in \mathbb{R}^d$

Robust regression

Assume different distribution of noise

$$y = x^{\mathrm{T}}\beta + \varepsilon, \varepsilon \sim \mathrm{Laplace}(0, \sigma).$$

Robust regression

Assume different distribution of noise

$$y = x^{\mathrm{T}}\beta + \varepsilon, \varepsilon \sim \mathrm{Laplace}(0, \sigma).$$

and obtain median absolute regression

$$\hat{\beta} = \arg\min_{\beta} \sum_i |x_i^{\mathrm{T}}\beta - y_i|.$$

Robust regression

Replace the square loss function by Huber or Hampel loss

$$\hat{\beta} = \arg\min_{\beta} \sum_i \rho(x_i^{\mathrm{T}} \beta - y_i)$$

Robust regression

Replace the mean estimate by robust alternatives

- least median of squares (LMS)

$$\hat{\beta} = \arg\min_{\beta} \operatorname{med}\left\{(x_i^{\mathrm{T}}\beta - y_i)^2\right\}$$

$\cdot_{(j)}$ represents an order statistics (quantile)

Robust regression

Replace the mean estimate by robust alternatives

- least median of squares (LMS)

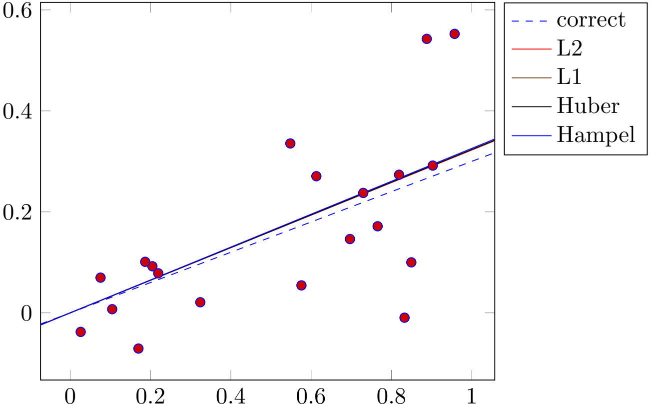
$$\hat{\beta} = \arg\min_{\beta} \operatorname{med} \left\{ (x_i^{\mathrm{T}} \beta - y_i)^2 \right\}$$

- least trimmed squares (LTS)

$$\hat{\beta} = \arg\min_{\beta} \sum_i (x_i^{\mathrm{T}} \beta - y_i)^2_{(j)}$$

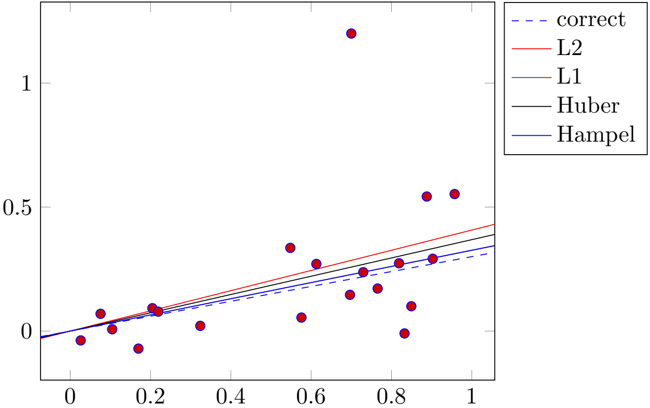
$\cdot_{(j)}$ represents an order statistics (quantile)

Examples of robust regression



$y = 0.1x + \mathcal{N}(0, 0.1)$ without outliers

Examples of robust regression



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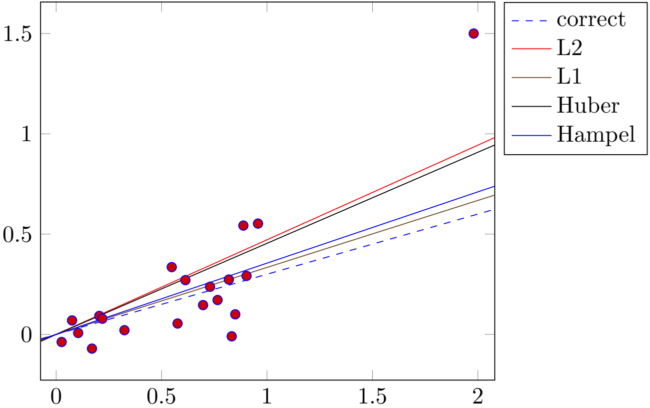
Examples of robust regression



$y = 0.1x + \mathcal{N}(0, 0.1)$ with outliers

Plan

Estimators of scale

- sample standard deviation
- median absolute deviation
- S_n
- Q

Sample standard deviation

- (unbiassed) formula: $\sigma^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$
- (biassed) formula: $\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$
- breakdown point 0
- ARE=1 — optimal for Normal distribution

$\bar{x}$ denotes mean of samples

Median absolute deviation

- formula: $\mathrm{MAD} = \mathrm{med}\{|x_i - \mathrm{med}\{x_i\}|\}$
- breakdown point 50%
- For Normal distribution
  - ARE=0.37
  - $\hat{\sigma} = 1.4826 \cdot \mathrm{MAD}$

S_n

- formula: $S_n = \mathrm{med}_i\{\mathrm{med}_j|x_i - x_j|\}$
- breakdown point 29%
- For Normal distribution
  - ARE=0.86
  - $\hat{\sigma} = 1.0483 \cdot S_n$

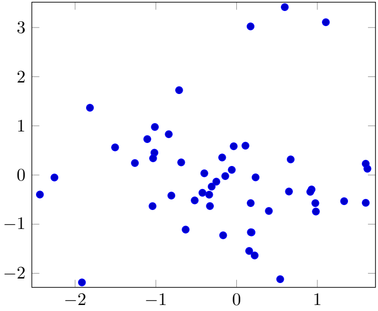
Q

- sample standard deviation $Q = \{|x_i - x_j||i < j\}_{q25}$
- breakdown point 50%
- For Normal distribution
  - ARE=0.82
  - $\hat{\sigma} = 2.2219 \cdot Q$

Plan

Pearson's correlation

- Assume pairs of samples $\{(x_i, y_i)\}_{i=1}^n$
- $\rho_{X,Y} = \frac{\frac{1}{n}\sum_i[(x_i - \bar{x})(y_i - \bar{y})]}{\sigma_X \sigma_Y}$



$\rho = -0.0008$

$\bar{x}$, $\bar{y}$ denotes a sample mean

Pearson's correlation

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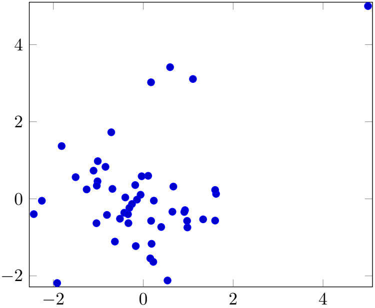
- $\rho_{X,Y} = \frac{\frac{1}{n}\sum_i[(x_i - \bar{x})(y_i - \bar{y})]}{\sigma_X \sigma_Y}$
- Breakdown point is zero

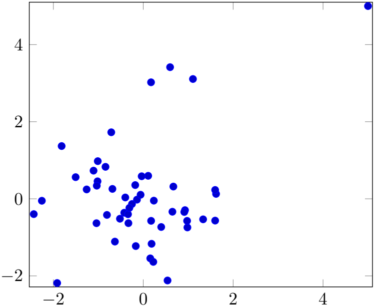


$\rho = 0.3245$

$\bar{x}$, $\bar{y}$ denotes a sample mean

Spearman's correlation

- Replaces $\{x_i, y_i\}_i$ by ranks $\{r_i^x, r_i^y\}_i$
- $r_s = \frac{\frac{1}{n}\sum_i[(r_i^x - \bar{r_x})(r_i^y - \bar{r_y})]}{\sigma_{r_x}\sigma_{r_y}}$

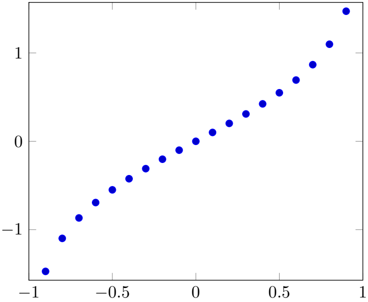


$r_s = -0.1025,\ \rho = 0.3245$

$\bar{r_x}$, $\bar{r_y}$ denotes a sample mean of ranks

Spearman's correlation

- Replaces $\{x_i, y_i\}_i$ by their ranks $\{r_i^x, r_i^y\}_i$
- $r_s = \frac{\frac{1}{n}\sum_i[(r_i^x - \bar{r_x})(r_i^y - \bar{r_y})]}{\sigma_{r_x}\sigma_{r_y}}$
- Statistic $r_s\sqrt{\frac{n-2}{1-r^2}}$ follows Student-t

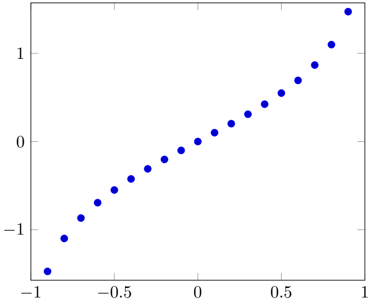


$r_s = 1.0,\ \rho = 0.98$

$\bar{r_x}$, $\bar{r_y}$ denotes a sample mean of ranks

Kendall correlation

- Kendalls'τ removes all quantities and uses order
- Samples are concordant if
  - $x_i < x_j$ and $y_i < y_j$
  - $x_i > x_j$ and $y_i > y_j$
- $r_k = \frac{1}{\binom{n}{2}}(n_c - n_d)$
- $\tau \sim \mathcal{N}\left(0, \frac{2(2N+5)}{9N(N-1)}\right)$



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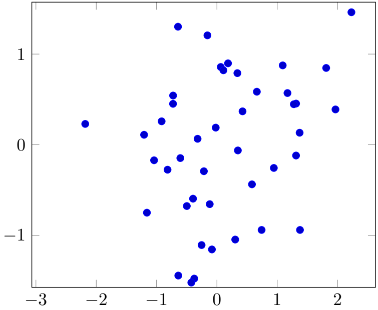

$r_k = 0.34,\ r_s = 0.48,\ \rho = 0.55$

Plan

Sign test

Tests if differences between pairs of observations are consistent.

Sign test

Population of pairs $\{(x_i, y_i)\}_i$

1. discard samples for which $|y_i - x_i| = 0$
2. test statistic
$$W = \sum_{i=1}^{N_r} I(y_i > x_i)$$
3. under null hypothesis W follows binomial distribution $\mathrm{Bi}(N, 0.5)$

$I(\cdot)$ denotes the indicator function

Wilcoxon-signed rank test

Tests if population of two related (matched) samples have equal mean rank.

Test hypothesis of Wilcoxon-signed rank test

Difference between pairs follows a symmetric distribution around zero.

Wilcoxon-signed rank test

Population of pairs $\{(x_i, y_i)\}_i$

1. calculate $|y_i - x_i|$ and discard those with $|y_i - x_i| = 0$
2. rank remaining samples according to $|y_i - x_i|$
3. test statistic
$$W = \sum_{i=1}^{N} [\mathrm{sgn}(y_i - x_i) \cdot R_i]$$
4. under null hypothesis W has
    - zero mean
    - variance $\sigma_w^2 = \frac{N(N+1)(2N+1)}{6}$
5. For small N critical values are tabulated.
6. For large N with $z = \frac{W}{\sigma_W}$

Discussion of sign and signed-rank test

- Sign test have less assumptions — needs only order relationship
- Signed rank test have higher power: ARE is 0.67.
- Would differences follows normal distribution, paired t-test is more appropriate; ARE is 0.95.
- Generalization of a sign test to *n*-tuples is a Friedman test.

Mann-Whitney U-test

Tests, whether a probability that a value from population X is greater than a value from population Y (and vice versa) is greater than 0.5.

Tests, whether the distributions of both populations are equal.

Mann-Whitney U-test

1. Calculate ranks of all samples together.
2. Sum ranks of samples, R_1 from first the population.
3. Calculate $U_1 = R_1 - \frac{n_1(n_1+1)}{2}$ and $U_2 = R_2 - \frac{n_2(n_2+1)}{2}$.
4. $U = \min\{U_1, U_2\}$
5. For small n_1, n_2 critical values are tabulated,
   for large n_1, n_2 $U \sim \mathcal{N}\left(\frac{n_1 n_2}{2}, \frac{n_1 n_2(n_1+n_2+1)}{12}\right)$.