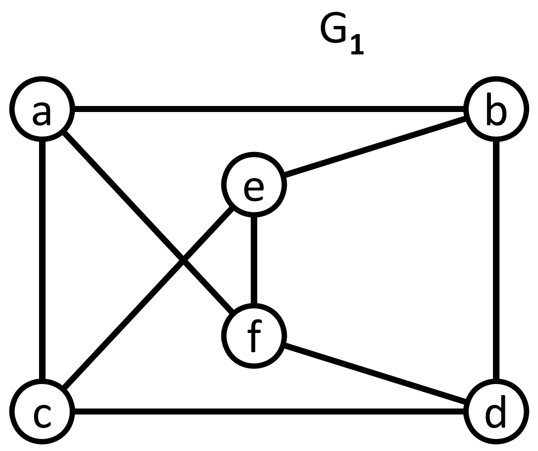
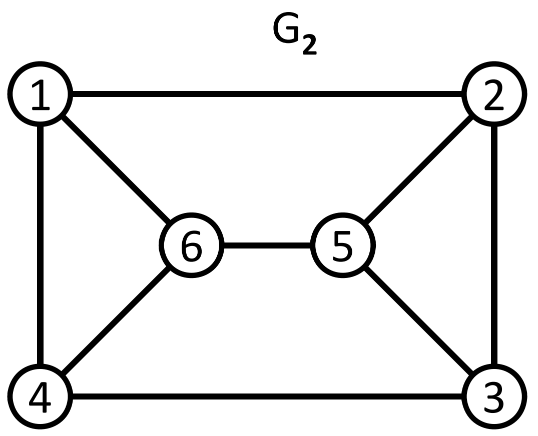
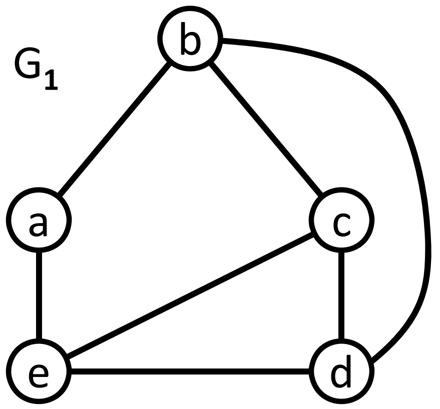
Examples of isomorphic and non-isomorphic graphs

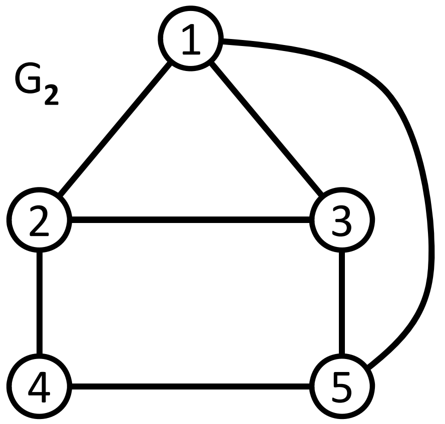




$\lvert V(G_1)\rvert = 6$	$\lvert V(G_2)\rvert = 6$
$\lvert E(G_1)\rvert = 9$	$\lvert E(G_2)\rvert = 9$
is regular = true	is regular = true
max degree = 3	max degree = 3
diameter = 2	diameter = 2
no. of triangles = 0	no. of triangles = 2 (triangles 1-4-6 and 2-3-5)

Examples of isomorphic and non-isomorphic graphs





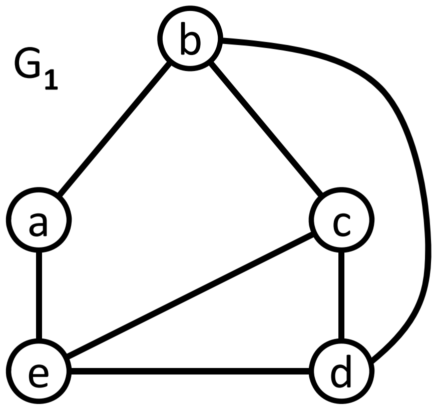
$\lvert V(G_1)\rvert = 5$	⟷	$\lvert V(G_2)\rvert = 5$
$\lvert E(G_1)\rvert = 6$	⟷	$\lvert E(G_2)\rvert = 6$
min degree = 2	⟷	min degree = 2
max degree = 3	⟷	max degree = 3
degree sequence = [3 3 3 3 2]	⟷	degree sequence = [3 3 3 3 2]
...		...
etc.		etc.

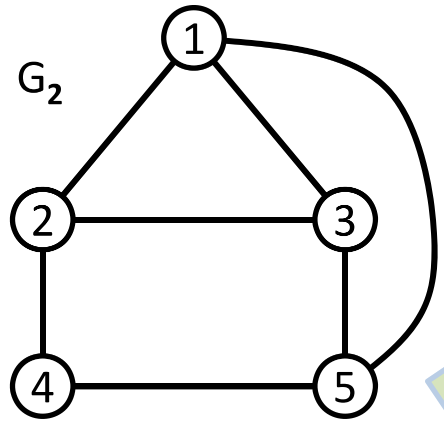
The question remains: ?

Are G_1 and G_2 isomorphic to each other?

?

Examples of isomorphic and non-isomorphic graphs





G_1: Set of edges:

{ {a e}
{a b}
{b c}
{c e}
{e d}
{c d}
{b d} }

Nodes mapping:

a --- 4
b --- 5
c --- 3
d --- 1
e --- 2

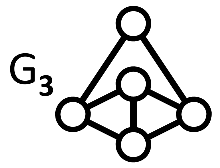
G_1: Set of mapped edges:

{ {4 2}
{4 5}
{5 3}
{3 2}
{2 1}
{3 1}
{5 1} }

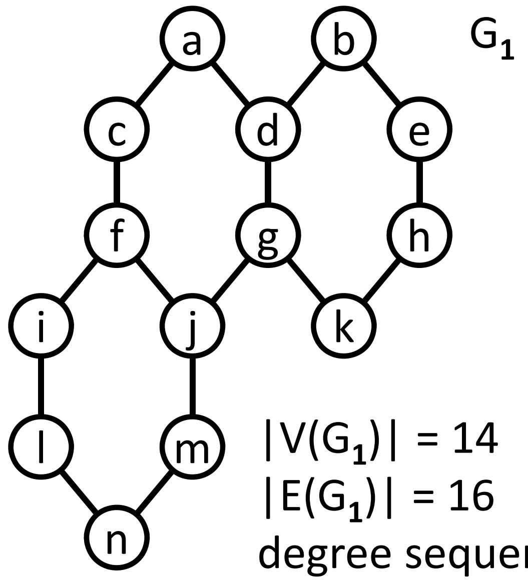
G_2: Set of edges:

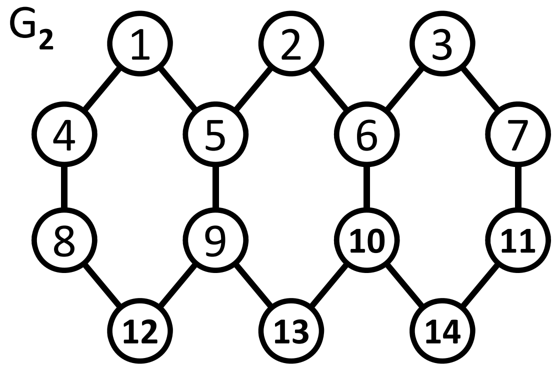
{ {1 2}
{1 3}
{1 5}
{2 3}
{2 4}
{3 5}
{4 5} }

Both sets of edges are the same. G_1 and G_2 are isomorphic.
(Verification: Sort all sets, compare items one by one.)



Examples of isomorphic and non-isomorphic graphs





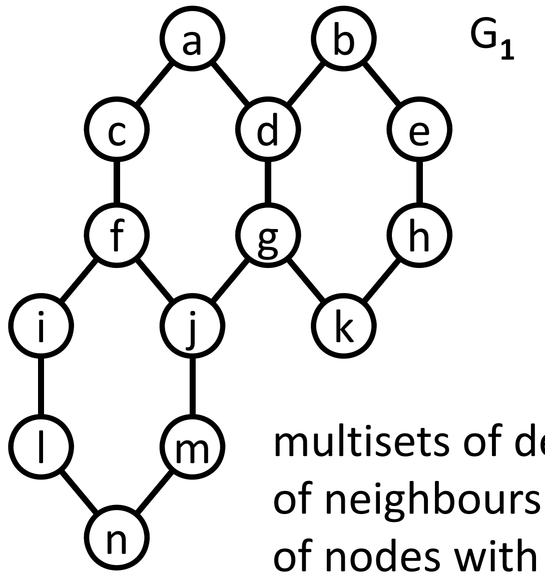
$\|V(G_1)\| = 14$	⟷	$\|V(G_2)\| = 14$
$\|E(G_1)\| = 16$	⟷	$\|E(G_2)\| = 16$
degree sequence = [3 3 3 3 2 2 2 2 2 2 2 2 2 2]	⟷	degree sequence = [3 3 3 3 2 2 2 2 2 2 2 2 2 2]
diameter = 7, (distance(l, e))	⟷	diameter = 7, (distance(4, 11))
isBipartite = yes	⟷	isBipartite = yes
...		...

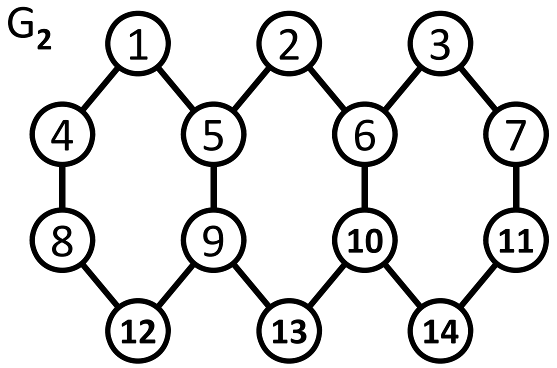
The question remains: ?

Are G_1 and G_2 isomorphic to each other?

?

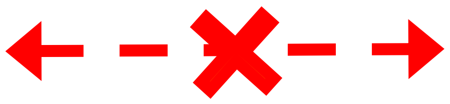
Examples of isomorphic and non-isomorphic graphs





multisets of degrees
of neighbours
of nodes with degree 3:

```
{ {3 2 2}    // d
  {3 2 2}    // f
  {3 3 2}    // g
  {3 3 2} }  // j
```



multisets of degrees
of neighbours
of nodes with degree 3:

```
{ {3 2 2}    // 5
  {3 2 2}    // 6
  {3 2 2}    // 9
  {3 2 2} }  // 10
```

G_1 and G_2 are not isomorphic to each other.

Another Invariant:

G_1 -- nodes of degree 3
form a connected subgraph.

G_2 -- nodes of degree 3
form two mutually unconnected subgraphs.

More invariants: Try yourself....

Difficulty

Is there any set of properties which are (relatively) easy to calculate for any graph and which values would decide whether two given graphs G_1, G_2 are isomorphic?
In the sense:

values calculated on G_1 == values calculated on G_2
if and only if
G_1 is isomorphic to G_2

So far, no such set of properties is known in general.

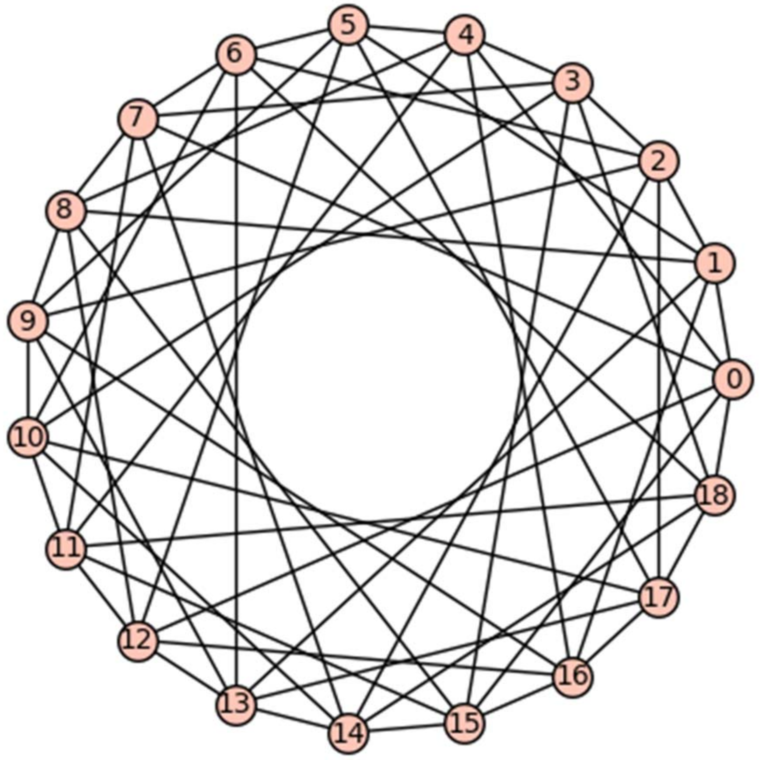
Partial solution

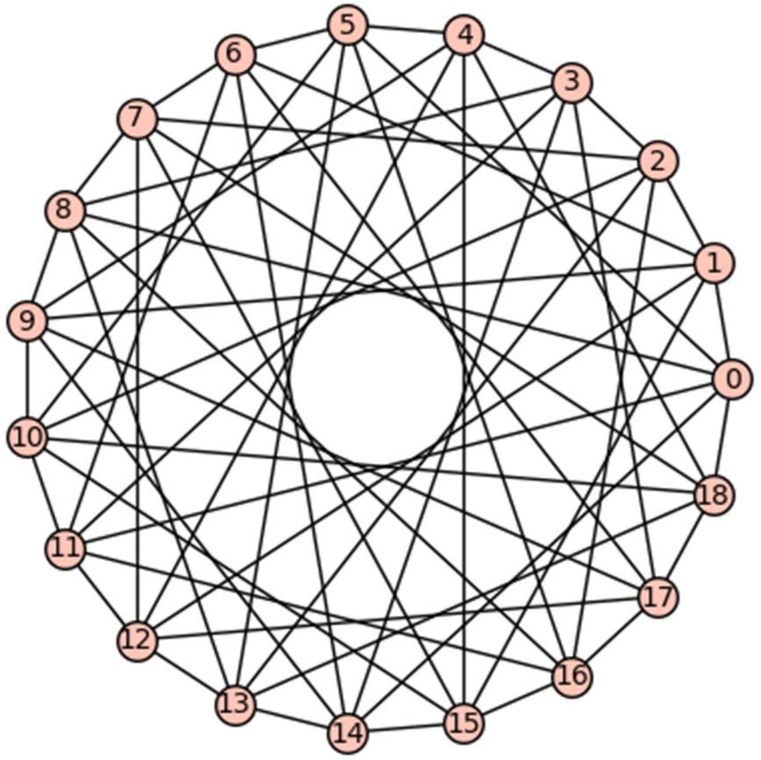
Advanced heuristical approaches solve the problem in many practical settings:
SW: **nauty** and **Traces:** https://pallini.di.uniroma1.it/

based on papers by Brendan D.McKay and AdolfoPiperno: *Practical graph isomorphism I and II*.
http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.169.6684
https://arxiv.org/abs/1301.1493

Examples of isomorphic and non-isomorphic graphs

Isomorphism is difficult to confirm/reject when the graphs are highly symmetric.
Informally, symmetry means that a graph "looks the same" in the vicinity of each node.
The number of candidate bijections is then difficult to reduce when there are no obvious invariants which values would help to distinguish between different nodes.
As a simple example, consider the following pair of graphs.





Picture credit to https://sagecell.sagemath.org and code

```
g1 = graphs.CirculantGraph(19, [1,5,8] ); g1.show()
g2 = graphs.CirculantGraph(19, [1,4,7] ); g2.show()
```

Isomorphism of directed graphs

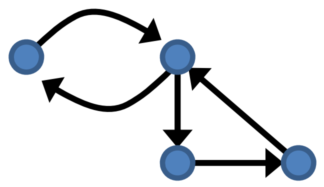
In these slides, term graph always refers to an undirected graph, if not specified otherwise.

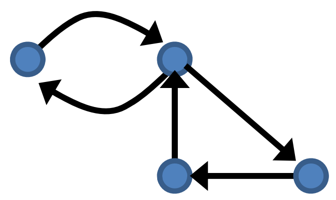
All isomorphism properties, algorithms, notions, etc. defined for undirected graphs, can be analogously defined and analyzed/solved in analogous manner for directed graphs.

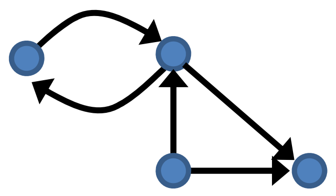
Two directed graphs G_1=$(V_1$,$E_1)$ and G_2=$(V_2$,$E_2)$ are *isomorphic* if there is a bijection $f: V_1 \to V_2$ such that

$$\forall\, x, y \in V_1 \quad : \quad (f(x), f(y)) \in E_2 \quad \Leftrightarrow \quad (x, y) \in E_1$$

Example:


G_1


G_2


G_3

Graphs G_1 and G_2 are isomorphic, G_3 is not isomorphic to any of G_1, G_2.

N	Number f(N) of graphs on N nodes (incl. unconnected ones) https://oeis.org/A000088
1	1
2	2
3	4
4	11
5	34
6	156
7	1044
8	12346
9	274668
10	12005168
15	31426485969804308768
20	645490122795799841856164638490742749440 ~ $6.5 \cdot 10^{38}$
30	334494316309257669249439569928080028956631479935393064329967834887217734534880582749030521599504384 ~ $3.3 \cdot 10^{98}$
40	779384116791497795458255081757517776606605527253316050186421058071969959228076659876210850745891393608193296535203737288659325928675388385701638330798186346244969194935885305312064818380 8 ~ $7.8 \cdot 10^{186}$
N	see inset

Approximation of the number of graphs: $$f(N) \le \frac{2^{\binom{N}{2}}}{N!}$$

The formula approximates f(N) tightly, in the sense: $$\lim_{N\to\infty} \frac{f(N)}{\frac{2^{\binom{N}{2}}}{N!}} = 1$$

Applying brute force and checking all graphs for would be a hopeless effort.

N	Number f'(N) of *connected* graphs on N nodes https://oeis.org/A001349
1	1
2	1
3	2
4	6
5	21
6	112
7	853
8	11117
9	261080
10	11716571
15	31397381142761241960
20	6454654831987227994267311287945022830 04
30	33449429761790292747406258898877142059240034044849717573548678757391976309266443346101758501370559 4
40	779384116734790137315958619064556399613117743568097366698224362707037749723541741787483239875824254167688055270461070798107972298831244753313320111264060419208367277602863359010916637465 9
N	asymptotically same as all graphs, in the sense: $\lim \{ N \to \infty , f'(N) / f(N) \} = 1$

Applying brute force and checking all graphs for would be a hopeless effort.

N	Number f''(N) of undirected trees on N nodes https://oeis.org/A001349
1	1
2	1
3	1
4	2
5	3
6	6
7	11
8	23
9	47
10	106
15	7741
20	823065
30	14830871802 ~ $1.5 \cdot 10^{10}$
40	363990257783343 ~ $3.6 \cdot 10^{14}$
100	630134658347465720563607281977639527019590
N	Formula is too complex to fit here, see the OEIS reference above

Applying brute force and checking all trees would be a hopeless effort.

Examples of more graph invariants (a tiny! selection):

(.) Connected - yes/no
(.) Number of edges
(.) Bipartite - yes/no
(.) Regular - yes/no (the degree of all nodes is the same)
(.) Tree - yes/no
(.) Planar - yes/no (can be drawn in a plane without edges crossing)
(X) Hamiltonian - yes/no (Hamilton path or cycle exists in the graph)
(.) Maximum/maximum node degree
(.) Number of nodes with maximum (minimum degree)
(.) Degree sequence (sequence of all node degrees sorted in non-increasing order)
(X) Spectrum (= multiset of eigenvalues) of adjacency (Laplacian) matrix of the graph
(X) Length of the shortest cycle (so called *girth* of the graph)
(X) Number of triangles
(.) Number of bridges/cutvertices/blocks
(X) Number of automorphisms
(X) Chromatic/independence/dominancy/clique numbers (see respective definitions...)
(X) Diameter, excentricity, number of centers
(X) Bandwidth
...
(.) *O(E+V),*   (X) *more complex than O(E+V), polynomial or exponential.*

Two random graphs are extremely(!) probably NOT isomorphic

When two graphs G1, G2 are selected randomly from the set of all graphs on N nodes or when they are generated randomly, then

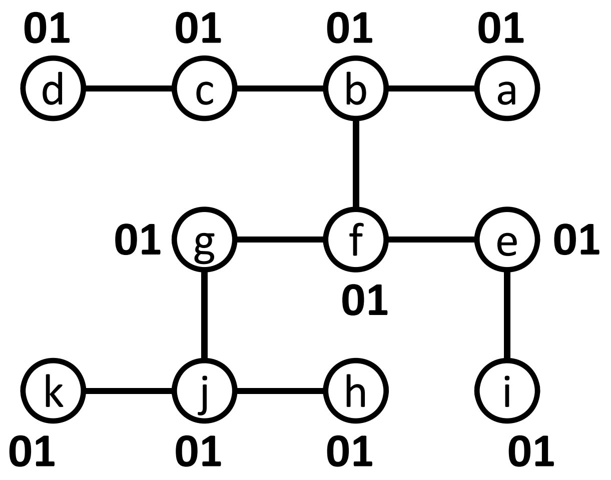
A. The probability that G1 and G2 are isomorphic is very close to 0. *)
B. The probability that the values of some (in fact, of many) of invariants in G1 and G2 are different is very close to 1.

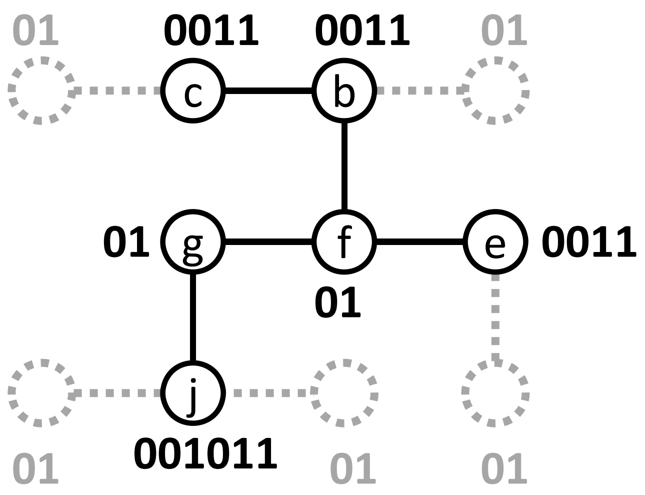
A. $\equiv$ Very probably, G1 and G2 are not isomorphic.
B. $\equiv$ Very probably, it is (relatively) easy to verify G1 and G2 are not isomorphic .

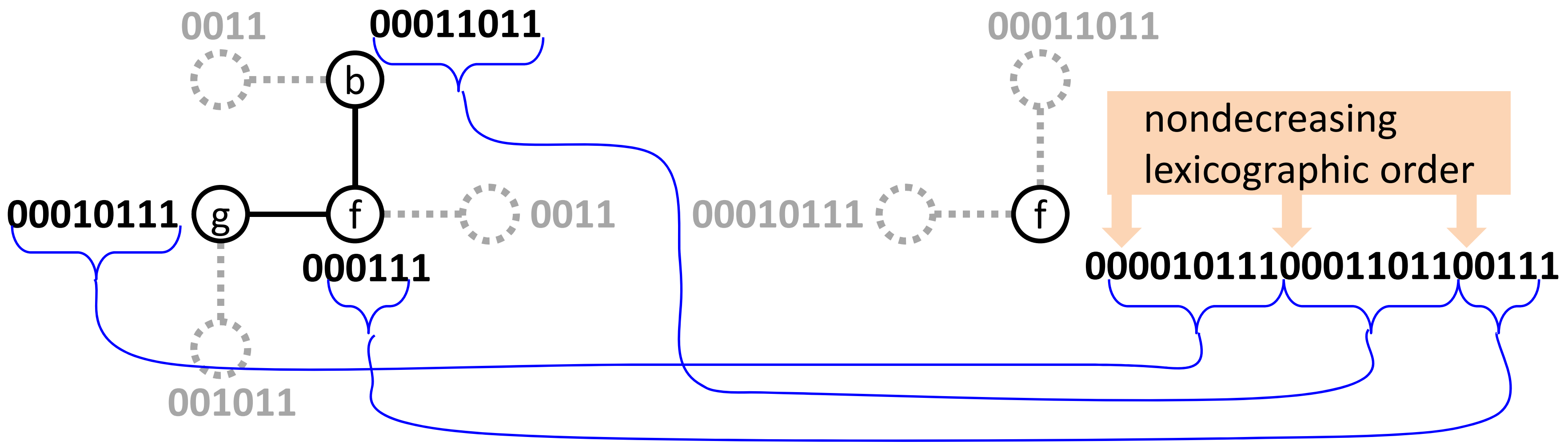
Conclusion:
When the graphs are not isomorphic,
checking the values of various (easy to compute, preferentially!) invariants in both graphs, quickly confirms the fact in majority of (random) cases.

*) How close? The probability p is in the order of $n! / 2^{\mathrm{comb}(n,2)}$.
For example, $n = 10$, $p = 10! / 2^{45} \cong 10^{-7}$; $n = 100$, $p = 100! / 2^{4950} \cong 10^{-1332}$.

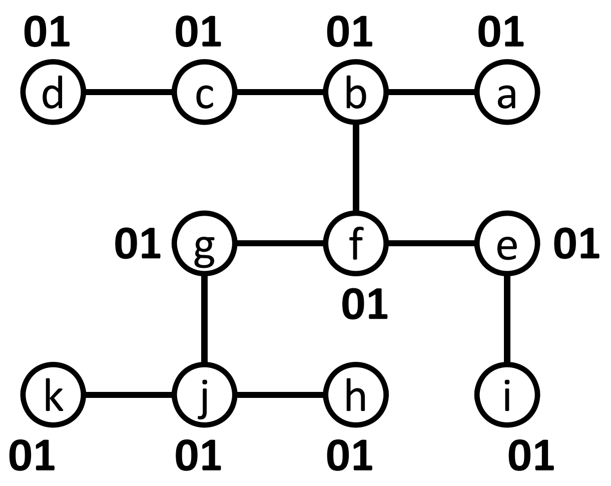
Tree certificate example



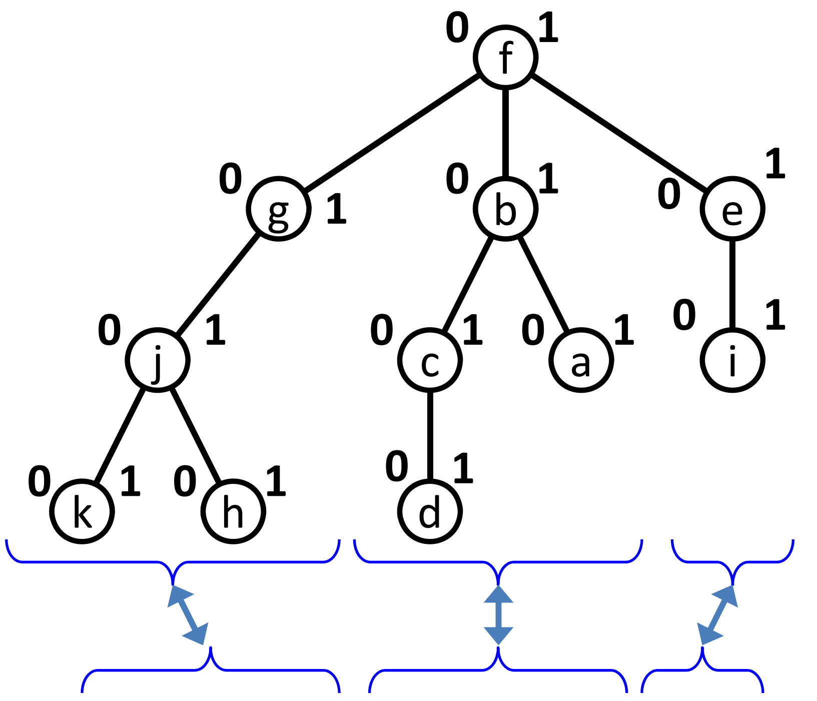




Tree certificate example

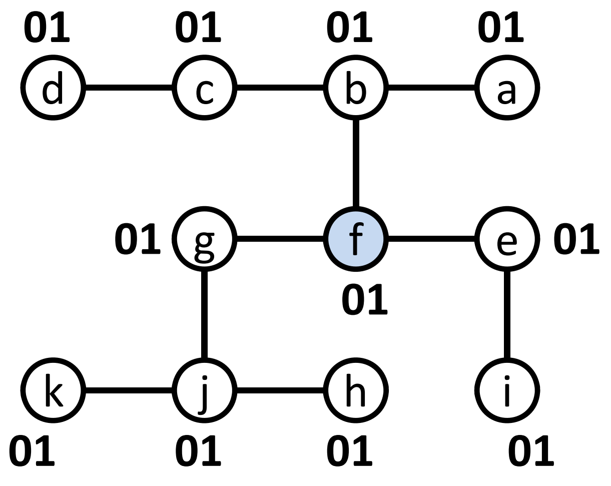


000010111000110110011

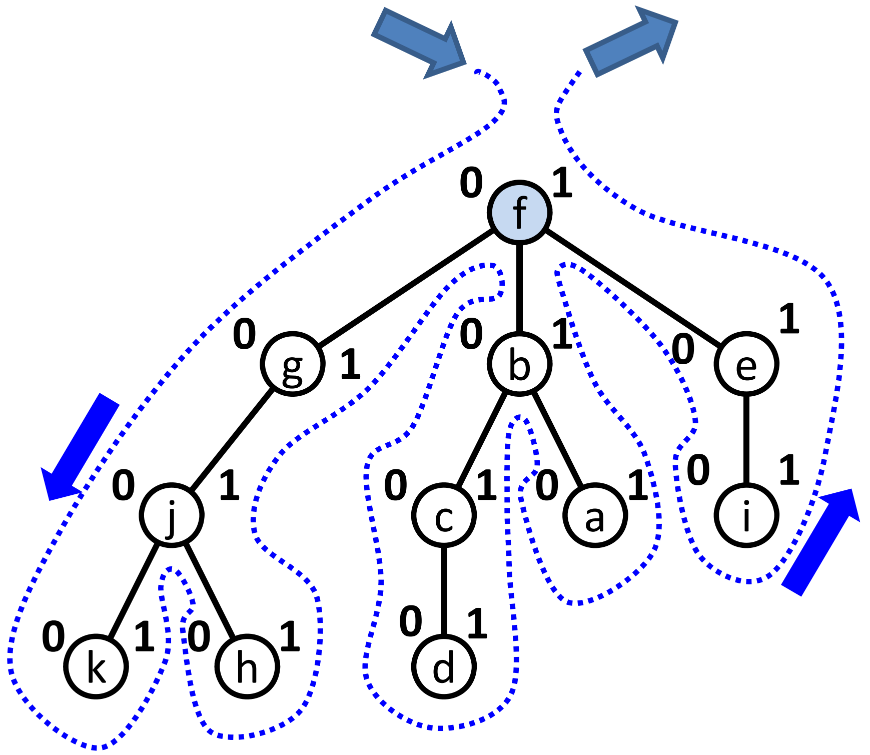


0 0 0 0 1 0 1 1 1 0 0 0 1 1 0 1 1 0 0 1 1 1

Tree certificate example



000010111000110110011



0 0 0 0 1 0 1 1 1 0 0 0 1 1 0 1 1 0 0 1 1 1

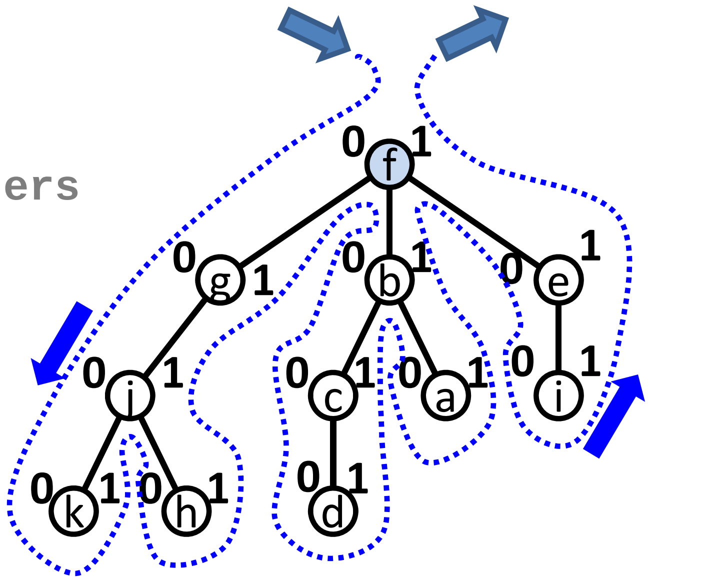
- Perform DFS from the root == center of the tree. Always expand DFS into that subtree which certificate is lexicographically the smallest.
- Output 0 when the node is being open and output 1 when the node is being closed.
- The output sequence is the tree certificate, it is obvious by induction.
- Drawback: DFS cannot know the subtrees certificates in advance.
- The idea can be used only for reconstructing the tree from the certificate.

Tree certificate example

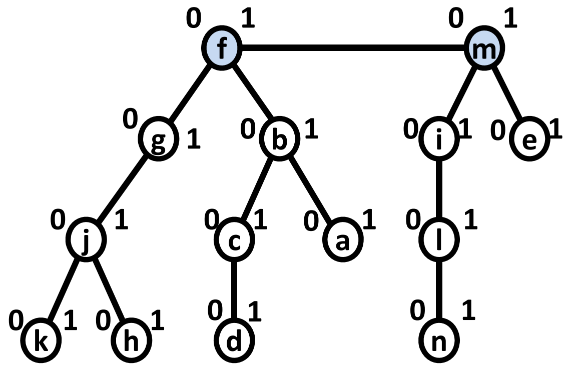
```
proc reconstructTree( certificate )
 nodesList = emptyList()
 edgesList = emptyList()
 centers = emptyList()  // one or two centers
 stack = emptyStack()

 for digit in certificate
   if digit == '0'
     create node X
     nodesList.add( X )
     if stack.isEmpty()
       centers.add( X )
     else
       edgesList.add( pair(stack.top(),X) )
     stack.push( X )
   else  // digit == '1'
     stack.pop()

 if centers.size() == 2  // two centers
   edgesList.add( pair(centers[0],centers[1]) )
return  nodesList, edgesList, centers
```



0 0 0 0 1 0 1 1 1 0 0 0 1 1 0 1 1 0 0 1 1 1



0000101110001101110000111011

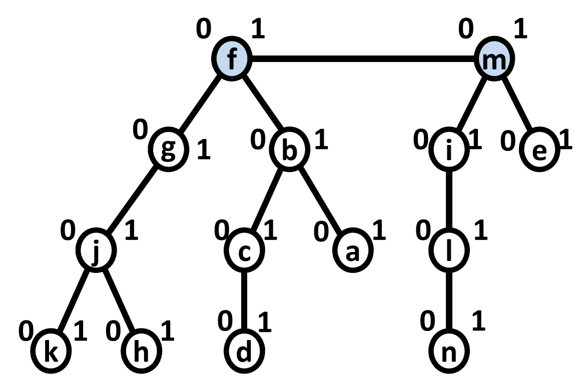
Tree certificate example - Python reconstruction

```
def reconstruct( certificate ):
   nodes, edges, stack  = [], [], []
   centers = [] # 1 or 2 centers
   newNode = 0  # nodes are integers

   for digit in certificate:
      if digit == '0':
         newNode += 1  # 'create' new node
         nodes.append( newNode )
         if len( stack ) == 0: # empty
            centers.append( newNode )
         else:
            edges.append( [newNode, stack[-1]] )
         stack.append( newNode )
      else: # digit == '1':
         stack.pop()

   if len( centers ) == 2:
      edges.append( [centers[0], centers[1]] )
   return nodes, edges, centers

cer = "0000101110001101110000111011"
nodes, edges, centers = reconstruct( cer )
```



0000101110001101110000111011