Constrained Least Squares

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Least squares with equality constraints

- the (linearly) *constrained least squares problem* (CLS) is

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|^2 \\ \text{subject to} & Cx = d \end{array}$$

- variable (to be chosen/found) is n-vector x
- $m \times n$ matrix A, m-vector b, $p \times n$ matrix C, and p-vector d are *problem data* (*i.e.*, they are given)
- $\|Ax - b\|^2$ is the *objective function*
- $Cx = d$ are the *equality constraints*
- x is *feasible* if $Cx = d$
- $\hat{x}$ is a *solution* of CLS if $C\hat{x} = d$ and $\|A\hat{x} - b\|^2 \leq \|Ax - b\|^2$ holds for any n-vector x that satisfies $Cx = d$

Piecewise-polynomial fitting

- *piecewise-polynomial* $\hat{f}$ has form

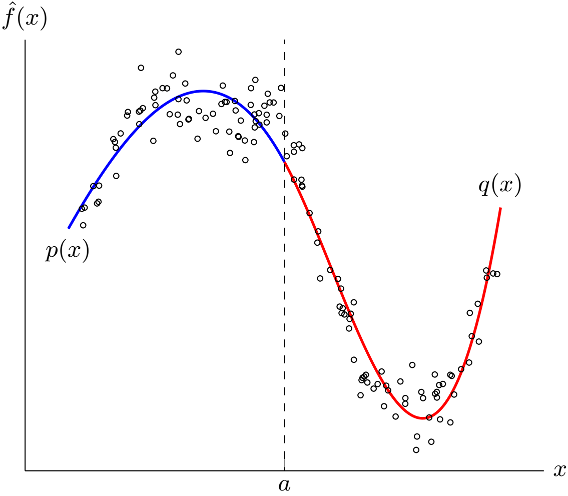
$$\hat{f}(x) = \begin{cases} p(x) = \theta_1 + \theta_2 x + \theta_3 x^2 + \theta_4 x^3 & x \leq a \\ q(x) = \theta_5 + \theta_6 x + \theta_7 x^2 + \theta_8 x^3 & x > a \end{cases}$$

  (a is given)
- we require $p(a) = q(a)$, $p'(a) = q'(a)$
- fit $\hat{f}$ to data (x_i, y_i), $i = 1, \ldots, N$ by minimizing sum square error

$$\sum_{i=1}^{N} (\hat{f}(x_i) - y_i)^2$$

- can express as a constrained least squares problem

Example



Piecewise-polynomial fitting

- constraints are (linear equations in θ)

$$\begin{array}{rcl} \theta_1 + \theta_2 a + \theta_3 a^2 + \theta_4 a^3 - \theta_5 - \theta_6 a - \theta_7 a^2 - \theta_8 a^3 & = & 0 \\ \theta_2 + 2\theta_3 a + 3\theta_4 a^2 - \theta_6 - 2\theta_7 a - 3\theta_8 a^2 & = & 0 \end{array}$$

- prediction error on (x_i, y_i) is $a_i^T \theta - y_i$, with

$$(a_i)_j = \begin{cases} (1, x_i, x_i^2, x_i^3, 0, 0, 0, 0) & x_i \leq a \\ (0, 0, 0, 0, 1, x_i, x_i^2, x_i^3) & x_i > a \end{cases}$$

- sum square error is $\|A\theta - y\|^2$, where a_i^T are rows of A

Least-norm problem

- special case of constrained least squares problem, with $A = I$, $b = 0$
- *least-norm problem*:

$$\begin{array}{ll} \text{minimize} & \|x\|^2 \\ \text{subject to} & Cx = d \end{array}$$

i.e., find the smallest vector that satisfies a set of linear equations

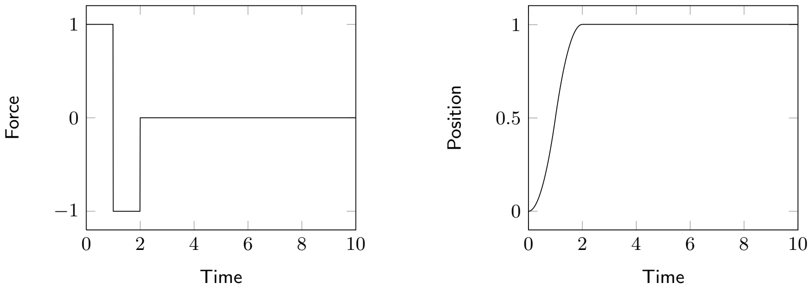
Force sequence

- unit mass on frictionless surface, initially at rest
- 10-vector f gives forces applied for one second each
- final velocity and position are

$$\begin{aligned} v^{\text{fin}} &= f_1 + f_2 + \cdots + f_{10} \\ p^{\text{fin}} &= (19/2)f_1 + (17/2)f_2 + \cdots + (1/2)f_{10} \end{aligned}$$

- let's find f for which $v^{\text{fin}} = 0$, $p^{\text{fin}} = 1$
- $f^{\text{bb}} = (1, -1, 0, \ldots, 0)$ works (called 'bang-bang')

Bang-bang force sequence



Least-norm force sequence

- let's find least-norm f that satisfies $p^{\text{fin}} = 1$, $v^{\text{fin}} = 0$
- least-norm problem:

$$\begin{array}{ll} \text{minimize} & \|f\|^2 \\ \text{subject to} & \left[\begin{array}{ccccc} 1 & 1 & \cdots & 1 & 1 \\ 19/2 & 17/2 & \cdots & 3/2 & 1/2 \end{array}\right] f = \left[\begin{array}{c} 0 \\ 1 \end{array}\right] \end{array}$$

  with variable f
- solution f^{ln} satisfies $\|f^{\text{ln}}\|^2 = 0.0121$ (compare to $\|f^{\text{bb}}\|^2 = 2$)

Least-norm force sequence