

Linear-Quadratic Control

Jenny Hong Nicholas Moehle Stephen Boyd

EE103
Stanford University

November 30, 2016

Linear dynamical system

$$x_{t+1} = Ax_t + Bu_t, \quad t = 1, 2, \dots$$

- ▶ n -vector x_t is *state* at time t
- ▶ m -vector u_t is *input* at time t
- ▶ $n \times n$ matrix A is *dynamics matrix*
- ▶ $n \times m$ matrix B is *input matrix*
- ▶ sequence x_1, x_2, \dots is called *state trajectory*

Simulation

- ▶ given x_1, u_1, u_2, \dots find x_2, x_3, \dots
- ▶ can be done by recursion: for $t = 1, 2, \dots$,

$$x_{t+1} = Ax_t + Bu_t$$

Vehicle example

consider a vehicle moving in a plane:

- ▶ sample position and velocity at times $\tau = 0, h, 2h, \dots$
- ▶ 2-vectors p_t and v_t are position and velocity at time ht
- ▶ 2-vector u_t gives applied force on the vehicle time ht
- ▶ friction force is $-\eta v_t$
- ▶ vehicle has mass m
- ▶ for small h ,

$$m \frac{v_{t+1} - v_t}{h} \approx -\eta v_t + u_t, \quad \frac{p_{t+1} - p_t}{h} \approx v_t$$

- ▶ we use approximate state update

$$v_{t+1} = (1 - h\eta/m)v_t + (h/m)u_t, \quad p_{t+1} = p_t + hv_t$$

- ▶ vehicle state is 4-vector $x_t = (p_t, v_t)$
- ▶ dynamics recursion is

$$x_{t+1} = Ax_t + Bu_t,$$

where

$$A = \begin{bmatrix} 1 & 0 & h & 0 \\ 0 & 1 & 0 & h \\ 0 & 0 & 1 - h\eta/m & 0 \\ 0 & 0 & 0 & 1 - h\eta/m \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ h/m & 0 \\ 0 & h/m \end{bmatrix}$$

Control

- ▶ x_1 is given
- ▶ choose u_1, u_2, \dots, u_{T-1} to achieve some goals, e.g.,
 - terminal state should have some fixed value: $x_T = x^{\text{des}}$
 - u_1, u_2, \dots, u_{T-1} should be small, say measured as

$$\|u_1\|^2 + \dots + \|u_{T-1}\|^2$$

(sometimes called 'energy')

- ▶ many control problems are linearly constrained least-squares problems

Minimum-energy state transfer

- ▶ given initial state x_1 and desired final state x^{des}
- ▶ choose u_1, \dots, u_{T-1} to minimize 'energy'

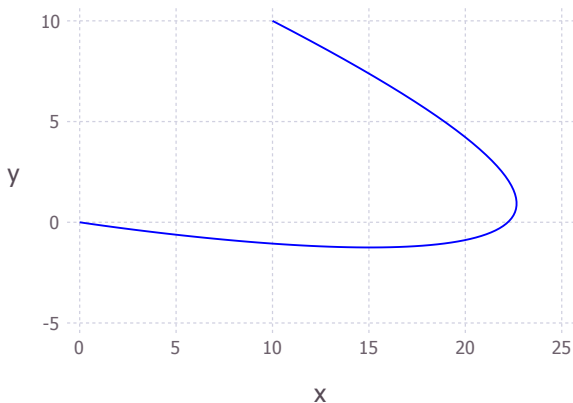
$$\begin{array}{ll} \text{minimize} & \|u_1\|^2 + \dots + \|u_{T-1}\|^2 \\ \text{subject to} & x_{t+1} = Ax_t + Bu_t, \quad t = 1, \dots, T-1 \\ & x_T = x^{\text{des}} \end{array}$$

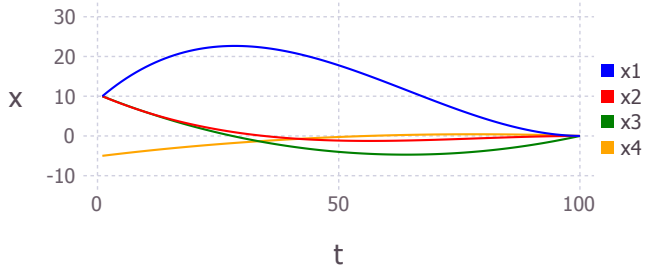
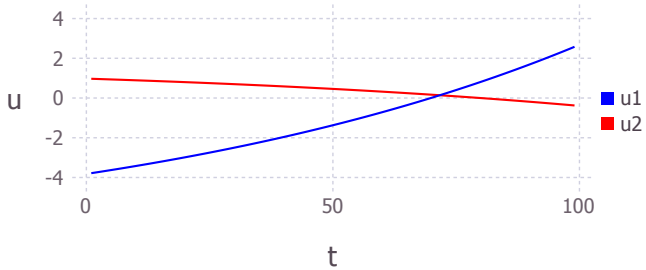
variables are $x_2, \dots, x_T, u_1, \dots, u_{T-1}$

- ▶ roughly speaking: find minimum energy inputs that steer the state to given target state over T periods

State transfer example

vehicle model with $T = 100$, $x_1 = (10, 10, 10, -5)$, $x^{\text{des}} = 0$





Output tracking

- ▶ $y_t = Cx_t$ is output (e.g., position)
- ▶ y_t should follow a desired trajectory, i.e., sum square *tracking error*

$$\|y_2 - y_2^{\text{des}}\|^2 + \dots + \|y_T - y_T^{\text{des}}\|^2$$

should be small

- ▶ the output tracking problem is

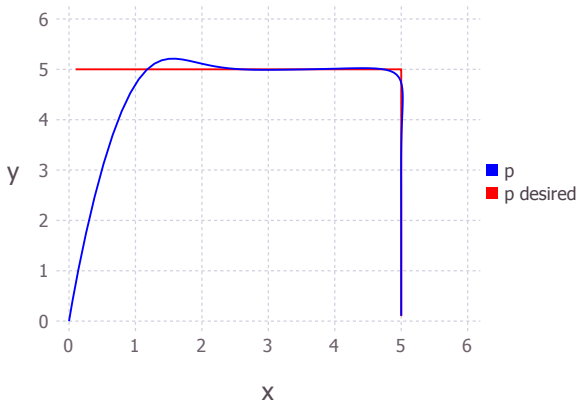
$$\begin{aligned} &\text{minimize} && \sum_{t=2}^T \|y_t - y_t^{\text{des}}\|^2 + \rho \sum_{t=1}^{T-1} \|u_t\|^2 \\ &\text{subject to} && x_{t+1} = Ax_t + Bu_t, \quad t = 1, \dots, T-1 \\ &&& y_t = Cx_t, \quad t = 1, \dots, T-1 \end{aligned}$$

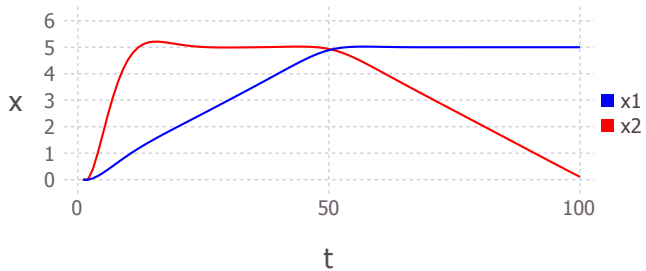
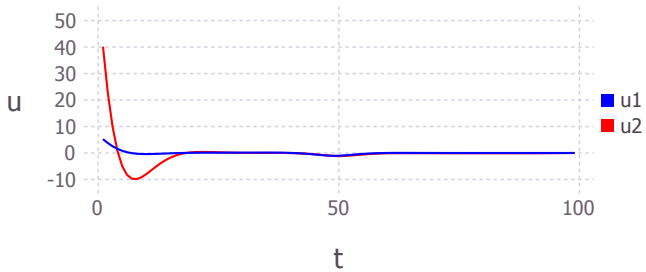
variables are $x_2, \dots, x_T, u_1, \dots, u_{T-1}, y_2, \dots, y_T$

- ▶ parameter $\rho > 0$ trades off control 'energy' and tracking error

Output tracking example

vehicle model with $T = 100$, $\rho = 0.1$, $x_1 = 0$, $y_t = p_t$ (position tracking)





Waypoints

- ▶ using output, can specify *waypoints*
- ▶ specify output (position) $w^{(k)}$ at time t_k at K total places

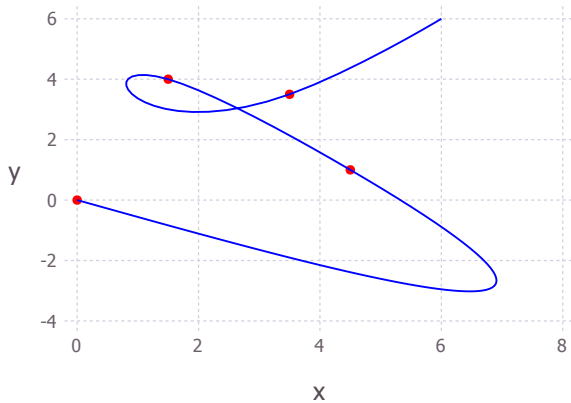
$$\begin{aligned} &\text{minimize} && \|u_1\|^2 + \dots + \|u_{T-1}\|^2 \\ &\text{subject to} && x_{t+1} = Ax_t + Bu_t, \quad t = 1, \dots, T-1 \\ &&& Cx_{t_k} = w^{(k)}, \quad k = 1, \dots, K \end{aligned}$$

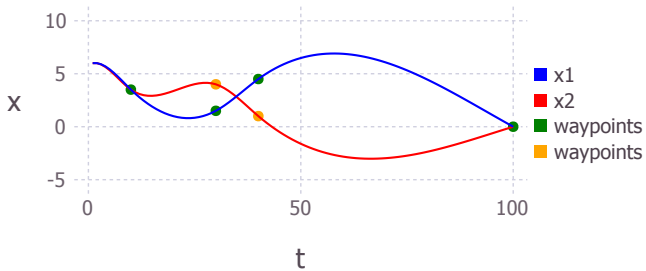
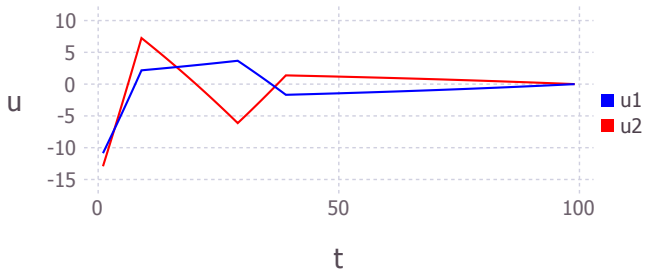
variables are $x_2, \dots, x_T, u_1, \dots, u_{T-1}$

Waypoints example

- ▶ vehicle model
- ▶ $T = 100$, $x_1 = (10, 10, 20, 0)$, $x^{\text{des}} = 0$
- ▶ $K = 4$, $t_1 = 10$, $t_2 = 30$, $t_3 = 40$, $t_4 = 80$
- ▶ $w^{(1)} = \begin{bmatrix} 3.5 \\ 3.5 \end{bmatrix}$, $w^{(2)} = \begin{bmatrix} 1.5 \\ 4 \end{bmatrix}$, $w^{(3)} = \begin{bmatrix} 4.5 \\ 1 \end{bmatrix}$, $w^{(4)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Waypoints example





Rendezvous

- ▶ we control two vehicles with dynamics

$$x_{t+1} = Ax_t + Bu_t, \quad z_{t+1} = Az_t + Bv_t$$

- ▶ final relative state constraint $x_T = z_T$
- ▶ formulate as state transfer problem:

$$\begin{aligned} & \text{minimize} && \sum_{t=1}^{T-1} (\|u_t\|^2 + \|v_t\|^2) \\ & \text{subject to} && x_{t+1} = Ax_t + Bu_t, \quad t = 1, \dots, T-1, \\ & && z_{t+1} = Az_t + Bv_t, \quad t = 1, \dots, T-1, \\ & && x_T = z_T \end{aligned}$$

variables are $x_2, \dots, x_T, u_1, \dots, u_{T-1}, z_2, \dots, z_T, v_1, \dots, v_{T-1}$

Rendezvous example

$$x_1 = (0, 0, 0, -5), \quad z_1 = (10, 10, 5, 0)$$

