by either deleting a single element or adding a single element. Such an ordering of the 2^n subsets of an n-set will be called a *minimal change ordering*.

As an example in the case $n = 3$, the ordering

$$\emptyset, \{3\}, \{2,3\}, \{2\}, \{1,2\}, \{1,2,3\}, \{1,3\}, \{1\}$$

is a minimal change ordering.

The characteristic vectors of the subsets in a minimal change ordering form a structure that is known as a *Gray code*. Thus, a Gray code is an ordering of the 2^n binary vectors of length n in such a way that any two consecutive vectors have Hamming distance equal to one.

From the minimal change ordering presented above, the following Gray code is obtained:

$$000, 001, 011, 010, 110, 111, 101, 100.$$

There is another way to formulate the concept of minimal change coverings or Gray codes. Consider the n-dimensional unit cube, whose 2^n vertices are labeled by the 2^n binary vectors. The edges of this cube join vertices having Hamming distance equal to one. Thus, a Gray code is nothing more than a *Hamiltonian path* in the n-dimensional unit cube, i.e., a method traversing the edges of the cube so that each vertex is visited exactly once. Examples are given in Figure 2.1.

There has been a considerable amount of study done on different constructions for Gray codes. We will look at a particularly nice class of Gray codes called the *binary reflected Gray codes*. G^n will denote the binary reflected Gray code for the 2^n binary n-tuples, and it will be written as a list of 2^n vectors, as follows:

$$G^n = [G^n_0, G^n_1, \ldots, G^n_{2^n-1}].$$

The codes G^n are defined recursively. The first one, G^1, is defined to be

$$G^1 = [0, 1].$$

Given G^{n-1}, the Gray code G^n is defined to be

$$G^n = \left[0G^{n-1}_0, \ldots, 0G^{n-1}_{2^{n-1}-1}, 1G^{n-1}_{2^{n-1}-1}, \ldots, 1G^{n-1}_0\right].$$

Equivalently, we have that

$$G^n_i = \begin{cases} 0G^{n-1}_i & \text{if } 0 \le i \le 2^{n-1} - 1 \\ 1G^{n-1}_{2^n-1-i} & \text{if } 2^{n-1} \le i \le 2^n - 1. \end{cases}$$

The code G^n is constructed from G^{n-1} in two steps. First, we take a copy of G^{n-1} with a "0" prepended to each vector. Then we take a copy of G^{n-1} in reverse order, with a "1" prepended to each vector. The fact that the second copy of G^{n-1} is in reverse order is the reason for the name "reflected."

The next two Gray codes produced by this recipe are

$$G^2 = [00, 01, 11, 10]$$

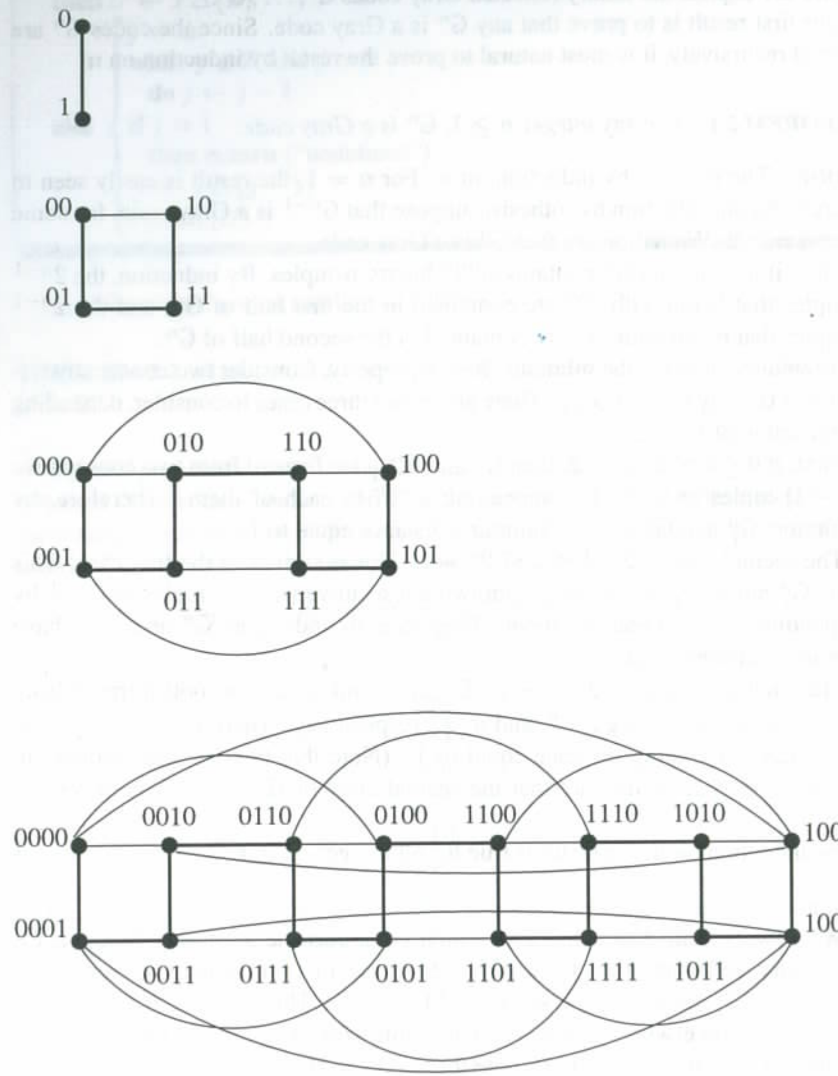

FIGURE 2.1
The evolution of the binary reflected Gray code.