Homework No. 02

The lecture discussed problems with non-local (spatially dispersive) material parameters, i.e., material relations of the following kind

$$\boldsymbol{D}(\boldsymbol{r},\omega)=\int_V \boldsymbol{\epsilon}(\boldsymbol{r}',\omega)\cdot\boldsymbol{E}(\boldsymbol{r}-\boldsymbol{r}',\omega)\,\mathrm{d}V' \tag{1}$$

or equivalently

$$\boldsymbol{D}(\boldsymbol{k},\omega)=\boldsymbol{\epsilon}(\boldsymbol{k},\omega)\cdot\boldsymbol{E}(\boldsymbol{k},\omega)\,. \tag{2}$$

Due to computational advantages, local material parameters

$$\boldsymbol{D}(\boldsymbol{r},\omega)=\boldsymbol{\epsilon}(\omega)\cdot\boldsymbol{E}(\boldsymbol{r},\omega) \tag{3}$$

are preferred whenever possible. It is therefore desired to have a simple test showing whether material parameters are local or not. A possible version of such test is developed in this homework.

Task No. 1: Assume source-less Maxwell's equations written in spectral domain

$$\begin{aligned}\mathrm{j}\boldsymbol{k}\times\boldsymbol{H}(\boldsymbol{k},\omega)&=\mathrm{j}\omega\boldsymbol{D}(\boldsymbol{k},\omega)\\ \mathrm{j}\boldsymbol{k}\times\boldsymbol{E}(\boldsymbol{k},\omega)&=-\mathrm{j}\omega\boldsymbol{B}(\boldsymbol{k},\omega)\\ \mathrm{j}\boldsymbol{k}\cdot\boldsymbol{B}(\boldsymbol{k},\omega)&=\boldsymbol{0}\\ \mathrm{j}\boldsymbol{k}\cdot\boldsymbol{D}(\boldsymbol{k},\omega)&=\boldsymbol{0}\end{aligned} \tag{4}$$

which describe propagation of planewaves with wavevector $\boldsymbol{k}$. Assume further linear local material relations

$$\begin{aligned}\boldsymbol{D}(\boldsymbol{k},\omega)&=\boldsymbol{\epsilon}(\omega)\cdot\boldsymbol{E}(\boldsymbol{k},\omega)+\frac{\boldsymbol{\chi}^{\mathrm{em}}(\omega)}{c_0}\cdot\boldsymbol{H}(\boldsymbol{k},\omega)\\ \boldsymbol{B}(\boldsymbol{k},\omega)&=\frac{\boldsymbol{\chi}^{\mathrm{me}}(\omega)}{c_0}\cdot\boldsymbol{E}(\boldsymbol{k},\omega)+\boldsymbol{\mu}(\omega)\cdot\boldsymbol{H}(\boldsymbol{k},\omega)\end{aligned} \tag{5}$$

where materials are described by second order tensors. Show that equation systems (4) and (5) can be rewritten into an eigenvalue equation

$$\begin{bmatrix}\boldsymbol{X}+\boldsymbol{\chi}^{\mathrm{me}} & \boldsymbol{\mu}_{\mathrm{r}}\\ -\boldsymbol{\epsilon}_{\mathrm{r}} & \boldsymbol{X}-\boldsymbol{\chi}^{\mathrm{em}}\end{bmatrix}\begin{bmatrix}\boldsymbol{E}\\ Z_0\boldsymbol{H}\end{bmatrix}=k_{\mathrm{n},z}\begin{bmatrix}\boldsymbol{Y} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{Y}\end{bmatrix}\begin{bmatrix}\boldsymbol{E}\\ Z_0\boldsymbol{H}\end{bmatrix} \tag{6}$$

where

$$\boldsymbol{X}=\begin{bmatrix}0 & 0 & k_{\mathrm{n},y}\\ 0 & 0 & -k_{\mathrm{n},x}\\ -k_{\mathrm{n},y} & k_{\mathrm{n},x} & 0\end{bmatrix}\quad \boldsymbol{Y}=\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \tag{7}$$

and where $\boldsymbol{k}_{\mathrm{n}}=c_0\boldsymbol{k}/\omega$, $\boldsymbol{\epsilon}_{\mathrm{r}}=\boldsymbol{\epsilon}/\epsilon_0$ $\boldsymbol{\mu}_{\mathrm{r}}=\boldsymbol{\mu}/\epsilon_0$ and $Z_0=\sqrt{\mu_0/\epsilon_0}$.

Task No. 2: Assume loss-less medium with $\boldsymbol{\epsilon}=\boldsymbol{\epsilon}^{\mathrm{H}}$, $\boldsymbol{\mu}=\boldsymbol{\mu}^{\mathrm{H}}$, $\boldsymbol{\chi}^{\mathrm{em}}=(\boldsymbol{\chi}^{\mathrm{me}})^{\mathrm{H}}$, where H denotes Hermitian conjugate and argue that, for chosen $k_{\mathrm{n},x},k_{\mathrm{n},y},\omega$, the system (6) gives six eigenvalues $k_{\mathrm{n},z}$ and corresponding six eigenwectors of planewaves propagating in the medium. Show that two of those eigenvalues are infinite and therefore do not represent physical solutions. We are thus left with four possible planewaves. Show that in the most important case of reciprocal medium ($\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\mathrm{T}}$, $\boldsymbol{\mu} = \boldsymbol{\mu}^{\mathrm{T}}$, $\boldsymbol{\chi}^{\mathrm{em}} = -\left(\boldsymbol{\chi}^{\mathrm{me}}\right)^{\mathrm{T}}$, where T denotes transposition) it holds that when $\boldsymbol{k}_{\mathrm{n}}$ is a solution, so is $-\boldsymbol{k}_{\mathrm{n}}$. Show at last that for an isotropic medium with no chirality ($\boldsymbol{\chi}^{\mathrm{em}} = \mathbf{0}$), these four solutions are, apart from sign, all identical.

Task No. 3: An important aspect of the eigensystem (6) is that each eigenvalue represents a relation $\omega\left(\boldsymbol{k}_{\mathrm{n}}\right)$, which is called dispersion relation. For fixed $|\boldsymbol{k}|$ this relation represents a surface, which is called isofrequency surface. Try to show (at least graphically) that in the local loss-less and reciprocal medium, these isofrequency surfaces are of two types: ellipsoids (so called definite media) or hyperboloids (so called indefinite media). This holds irrespective of how complex the medium tensors are. This is very strong constraint that distinguishes the local medium from a non-local one. In the non-local medium, the material parameters depend on $\boldsymbol{k}$ and (6) is a transcendental equation system which can have solutions as exotic as we can imagine.