# Version Space

## Theorem 1

*A finite hypothesis class $\mathcal{H}$ is learnable online from $X$ if $\lg|\mathcal{H}| \leq poly(n_X)$.*

Proof: the **version space** algorithm below has the mistake bound $\lg|\mathcal{H}|$ so if $\lg|\mathcal{H}| \leq poly(n_X)$ the version space learns $\mathcal{H}$ online from $X$.

Define a decision policy for a *set of hypotheses* as the *majority vote* among the hypotheses in it

$$\mathcal{H}(x) = \begin{cases} 1 \text{ if } |\{\, h \in \mathcal{H} \;:\; h(x) = 1 \,\}| > |\mathcal{H}|/2 \\ 0 \text{ otherwise} \end{cases} \tag{1}$$

Instead of a single hypothesis $h_k$, the version-space agent maintains a set $\mathcal{H}_k$ of hypotheses at each time $k \in \mathbb{N}$.

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# Version Space (cont'd)

At $\forall k \in \mathbb{N}$

$$y_{k+1} = \mathcal{H}_k(x_k)$$

where $\mathcal{H}_1 = \mathcal{H}$ and $\mathcal{H}_{k+1}$ keeps exactly those hypotheses from $\mathcal{H}_k$ which gave the correct decision for $x_k$:

$$\mathcal{H}_{k+1} = \{\, h \in \mathcal{H}_k : h(x_k) = \overline{y}_k \,\} \tag{2}$$

where $\overline{y}_k = |y_k + r_{k+1}|$ is the true class according to the target hypothesis.

At least half of the hypotheses from $\mathcal{H}_k$ is deleted by (2) on each mistake ($r_k = -1$), because at least half of them were wrong according to (1). In the worst case, the single remaining hypothesis is the target hypothesis, which will no longer make mistakes. So $\lg |\mathcal{H}_1| = \lg |\mathcal{H}|$ is the maximum number of mistakes.

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# Online Learnability due to Polynomial $\lg |\mathcal{H}|$

Theorem (1) implies online learnability of the hypothesis classes

$$\mathcal{H}_1 \subset \mathcal{H}_2 \subset \mathcal{H}_3$$

$\mathcal{H}_1$= monotone conjunctions (clauses, respectively): $|\mathcal{H}| = 2^n$ (each atom present or absent in it).

$\mathcal{H}_2$= contingent conjunctions (clauses, respectively): $|\mathcal{H}| = 3^n$ (each atom absent, present in a positive literal or present in a negative literal)

$\mathcal{H}_3$= conjunctions (clauses, respectively): $|\mathcal{H}| = 2^{2n}$ (each of $2n$ literals present or absent)

from $X = \{0,1\}^n$ when $n_X = n$ due to $\lg |\mathcal{H}| \leq poly(n)$.

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# Version Space vs. Algorithms for Specific $\mathcal{H}$

We already know these classes are *efficiently* learnable online from $X = \{0,1\}^n$ by Winnow with mistake bound $\mathcal{O}(\lg n)$. Version space has bound $\lg |\mathcal{H}| = \mathcal{O}(n)$ in all these cases and is *not efficient* as it loops over an exponential-size set $\mathcal{H}$ in (1) and (2).

Contingent conjunctions (clauses) are also *efficiently* learnable online from contingent conjunctions (clauses) by generalization with bound $\mathcal{O}(n)$ where $n_X = n$ is the maximum number of variables in the conjunctions (clauses). The version space bound $\lg |\mathcal{H}| = \mathcal{O}(n)$ is the same but again, version space is *not effificent*.

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# Online Learnability due to Polynomial $\lg|\mathcal{H}|$ (cont'd)

Further classes learnable online from $X = \{0,1\}^n$:

$\mathcal{H} =$ s-conjunctions or s-clauses: not only $\lg|\mathcal{H}| \leq poly(n)$ but $|\mathcal{H}| \leq poly(n)$. So version space learns them online with logarithmic mistake bound and *efficiently* because (1) and (2) can be evaluated in $poly(n)$-time.

$\mathcal{H} =$ s-DNF or s-CNF: These include a subset of the $poly(n)$-number of s-conjunctions (s-clauses, respectively), so $|\mathcal{H}| = 2^{poly(n)}$, i.e. $\lg|\mathcal{H}| = poly(n)$. So version space learns them online with logarithmic mistake bound but not efficiently. In contrast, they can be learned online efficiently by reduction to monotone disjunctions (conjunctions).

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# VC Dimension

We say that concept class $\mathcal{C}$ **shatters** a set of observations $X' \subseteq X$ if for every subset $X'' \subseteq X'$ there is a concept $C \in \mathcal{C}$ such that $C \cap X' = X''$.

In other words, $X'$ is shattered by $\mathcal{C}$ if it can be split by concepts from $\mathcal{C}$ in all $2^{|X'|}$ possible ways.

## Vapnik-Chervonenkis Dimension

The **VC-dimension of concept class $\mathcal{C}$ on $X$** denoted $\mathrm{VC}(\mathcal{C})$ is

$$\max\{\ |X'|\ :\ \mathcal{C} \text{ shatters } X',\ X' \subseteq X\ \}$$

For a hypothesis class $\mathcal{H}$, we abbreviate $\mathrm{VC}(\mathcal{C}(\mathcal{H}))$ as $\mathrm{VC}(\mathcal{H})$ and call the latter the VC-dimension of $\mathcal{H}$.

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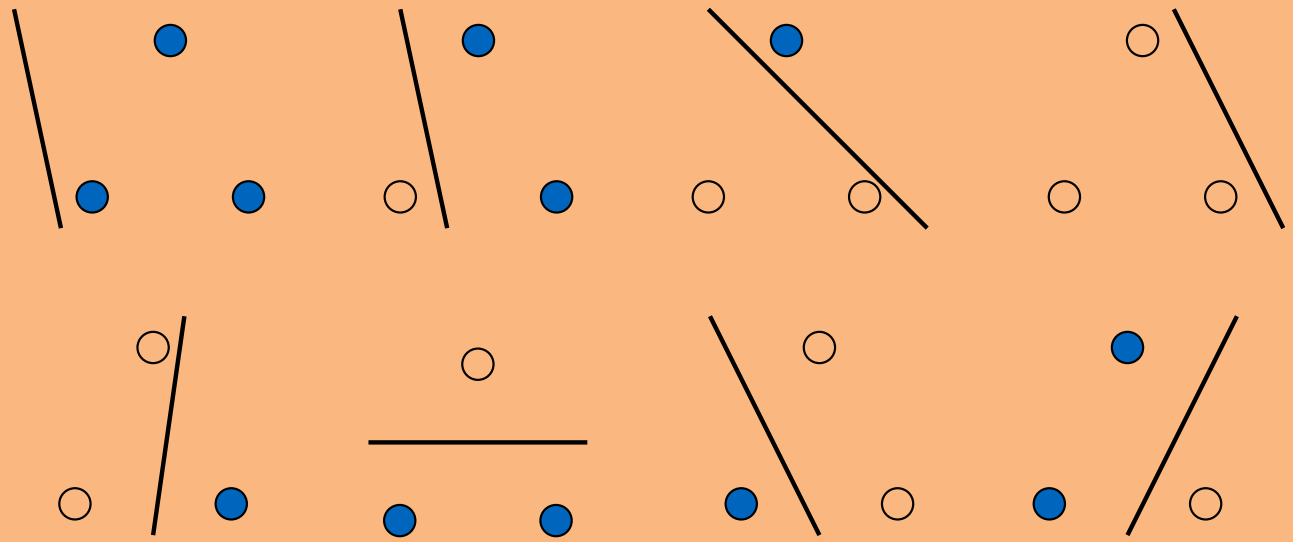
# Example: Determining VC-Dimension

- If *some* $X' \subseteq X$ shattered by $\mathcal{C}$ then $\mathrm{VC}(\mathcal{C}) \geq |X'|$.
- If *none* $X' \subseteq X$ shattered by $\mathcal{C}$ then $\mathrm{VC}(\mathcal{C}) < |X'|$.

Example: $\mathcal{C}$ = half-planes in $\mathbb{R}^2$ (i.e., linear separation)

- *Some* 3 points can be shattered



so $\mathrm{VC}(\mathcal{C}) \geq 3$.

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# Determining VC-Dimension (cont'd)

- *No* 4 points can be shattered. Obvious if 3 in line. Otherwise two cases possible:

One point in the middle  No point in the middle

In both cases, the colored subset cannot be separated by a line. So $\mathrm{VC}(\mathcal{C}) < 4$

We have $\mathrm{VC}(\mathcal{C}) \geq 3$ and $\mathrm{VC}(\mathcal{C}) < 4$, thus $\mathrm{VC}(\mathcal{C}) = 3$.

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# Poly VC-Dimension Necessary for Learnability

**Theorem 2**

*Concept class $\mathcal{C}$ on $X$ is learnable online from $X$ only if $\mathrm{VC}(\mathcal{C}) \leq poly(n_X)$.*

Proof: There exists a set of $\mathrm{VC}(\mathcal{C})$ observations from $X$ shattered by $\mathcal{C}$ so there exists a sequence $x_{\leq \mathrm{VC}(\mathcal{C})}$ of observations such that for any sequence of the learner's decisions $y_2, y_3, \ldots, y_{\leq \mathrm{VC}(\mathcal{C})+1}$ there is a target concept $C \in \mathcal{C}$ on which this sequence results in $\mathrm{VC}(\mathcal{C})$ mistakes, i.e., $r_2, r_3, \ldots, r_{\mathrm{VC}(\mathcal{C})+1} = \underbrace{(-1, -1, \ldots -1)}_{\mathrm{VC}(\mathcal{C})}$, thus $\sum_{k=1}^{\infty} |r_k| \geq \mathrm{VC}(\mathcal{C})$. So if $\mathcal{C}$ is learnable, i.e. if $\sum_{k=1}^{\infty} |r_k|$ is polynomially bounded, $\mathrm{VC}(\mathcal{C})$ is also polynomially bounded.

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# $\lg|\mathcal{H}|$ vs. $\mathrm{VC}(\mathcal{H})$

Theorem (2) holds for any concept class $\mathcal{C}$, including a concept class $\mathcal{C}(\mathcal{H})$ induced by a hypothesis class $\mathcal{H}$. So if $\mathcal{H}$ is learnable online from $X$ then $\mathrm{VC}(\mathcal{H}) \leq poly(n_X)$.

Combining this with Theorem (1), we get that $\mathrm{VC}(\mathcal{H})$ must be polynomial whenever $\lg|\mathcal{H}|$ is.

Note that in Theorem (1), we cannot replace "hypothesis class" with "concept class" and $\mathcal{H}$ with $\mathcal{C}$, because there may be concepts in $\mathcal{C}$ that have no finite description by a hypothesis, thus the assumption in the Theorem would not be sufficient for online learnability.

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# Standard Agent

An agent is called **standard** if it changes its hypothesis iff an error is made, i.e. $h_{k+1} \neq h_k$ iff $r_{k+1} = -1$. So Winnow and the generalization agent are both standard while version space is not.

## Lemma 1

*Let $r_{\leq K}$ be a reward sequence of a standard agent and $h_{\leq K}$ be its sequence of hypotheses. Denote $M = \sum_{k=1}^{K} |r_k|$. There is a hypothesis $h$ retained for at least $\frac{K}{M+1}$ consecutive steps in $h_{\leq K}$, i.e.*

$$h_{\leq K} = h_1, h_2, \ldots \underbrace{h, h, \ldots h}_{\text{at least } \frac{K}{M+1} \text{ times}}, \ldots h_K$$

Proof: exercise

*If $\frac{K}{M+1}$ is non-integer than "at least $\frac{K}{M+1}$" means the same as "at least the nearest integer largest than $\frac{K}{M+1}$".*

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# I.I.D. Observations and Hypothesis Error

Assume (??) for the rest of the concept-learning chapter.

Given target concept $\overline{C}$, define the **error of hypothesis** $h$ as

$$\text{err}(h) = P(\overline{C} \triangle C(h))$$

where $\triangle$ is the symmetric set difference. *Recall the definition of C(h).*

So err($h$) is the

- total probability of the error region on $X$ with respect to the distribution (??).
- probability that policy $h(x)$ is not the true class of $x$, i.e., not the optimal action (??).

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# The PAC Learning Model

Unlike the mistake bound model, the PAC-learning model does not require a finite bound on mistakes. It requires finding a *low-error hypothesis* with *high probability* using a *polynomial number of observations*.

## Probably Approximately Correct (PAC) Learning Model

In the concept classification protocol, an agent **probably approximately correctly (PAC) learns $\mathcal{C}$ from $X$** if for any target concept from $\mathcal{C}$ on $X$, an arbitrary distribution (??) and arbitrary numbers $0 < \epsilon, \delta < 1$, there is a $K < poly(n_X, 1/\epsilon, 1/\delta)$ such that with probability at least $1 - \delta$, the agent's hypothesis $h_K$ has $\text{err}(h_K) \leq \epsilon$. It PAC-learns $\mathcal{C}$ from $X$ **efficiently**, if in addition, the time taken to compute an action from an observation is also at most polynomial in $n_X, 1/\epsilon, 1/\delta$.

$\mathcal{C}$ is **PAC-learnable** if there is an algorithm that PAC-learns it.

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# Online Learnability Implies PAC-Learnability

Call a hypothesis $h$ **consistent** with $x \in X$ if $h(x)$ is the true class of $x$, i.e. $h(x) = 1$ iff $x$ is in the target concept.

## Theorem 3

*If $\mathcal{C}$ is learnable online from $X$ by a standard agent, then $\mathcal{C}$ is PAC-learnable from $X$.*

*The standard-agent assumption is not needed in the theorem but it is a rather weak assumption making the proof much easier.*

Proof: Online learnability of $\mathcal{C}$ from $X$ means $M = \sum_{k=1}^{\infty} |r_k| \leq poly(n_X)$. Due to Lemma (1), for an arbitrary $K \in \mathbb{N}$, some hypothesis $h$ was retained by the agent for $q = \frac{K}{M+1}$ steps before time $K$. Since the learning agent is standard, $h$ is consistent with $q$ consecutive observations.

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# Online Learnability Implies PAC-Learnability (cont'd)

To show that the agent PAC-learns $\mathcal{C}$, we want to show that $\text{err}(h) \leq \epsilon$ with probability at least $1 - \delta$ when the observations are i.i.d. from (??).

Call $h$ **bad** if $\text{err}(h) > \epsilon$. The probability that a $h$ is consistent with an observation from (??) is $1 - \text{err}(h)$, so if $h$ is bad, it is at most $(1 - \epsilon)$. Because $h$ is consistent with $q$ such i.i.d. consecutive observations, the probability that $h$ is bad is at most $(1 - \epsilon)^q$.

*Note that it is important for the above reasoning that h is consistent with q observations, which are consecutive. For example, we could not 'pick' an arbitrary selection of q correctly classified observation from an arbitrarily long history $x_{\leq k}$ (that would not be a series of Bernoulli trials).*

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# Online Learnability Implies PAC-Learnability (cont'd)

We want the probability $(1-\epsilon)^q$ to be less than $\delta$. Use the upper bound

$$(1-\epsilon)^q < e^{-\epsilon q} \tag{3}$$

which holds for all $\epsilon > 0$. So $(1-\epsilon)^q < \delta$ if $e^{-\epsilon q} \leq \delta$. The latter is satisfied if

$$q = \frac{1}{\epsilon} \ln \frac{1}{\delta} \tag{4}$$

because $e^{-\epsilon \frac{1}{\epsilon} \ln \frac{1}{\delta}} = \delta$. Since $q = \frac{K}{M+1}$, we have $K = (M+1)q$ where $M+1 \leq poly(n_X)$ and $q \leq poly(\frac{1}{\epsilon}, \frac{1}{\delta})$, therefore $K \leq poly(n_X, \frac{1}{\epsilon}, \frac{1}{\delta})$.

Thus with probability *at least* $\delta$, $h_K$ is *not bad* ($\text{err}(h) \leq \epsilon$) for $K \leq poly(n_X, \frac{1}{\epsilon}, \frac{1}{\delta})$ so the agent PAC-learns $\mathcal{C}$.

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# PAC-Learnability Implies Consistent Learning

The following theorem states that in the definition of PAC-Learning, the hypothesis $h_K$ is necessarily consistent with all observations up to $K-1$.

## Theorem 4

*Let an agent PAC-learn $\mathcal{C}$ from $X$. Then for any target concept from $\mathcal{C}$ on $X$, an arbitrary distribution (??) and arbitrary numbers $0 < \epsilon, \delta < 1$ and $K \in \mathbb{N}$, the condition $\mathrm{err}(h_K) \leq \epsilon$ with probability at least $1-\delta$ implies that $h_K$ is consistent with all observations in $x_{<K}$.*

Proof: exercise

# Consistent Agent

**$\mathcal{H}$-consistent learning agent** is an agent whose $h_{k+1}$ ($k \in \mathbb{N}$) is an arbitrary hypothesis from $\mathcal{H}$ that is consistent with $x_{\leq k}$ if such a hypothesis exists.

When the target concept is chosen from a concept class $\mathcal{C}$, the condition

$$\mathcal{C} \subseteq \mathcal{C}(\mathcal{H}) \tag{5}$$

(where $\mathcal{C}(\mathcal{H})$ is defined by (??)) guarantees that a hypothesis consistent with any $x_{\leq k}$ exists in $\mathcal{H}$.

Note that an $\mathcal{H}$-consistent agent may make mistakes even when (5) is satisfied but after each mistake ($r_{k+1} = -1$), $h_{k+1}$ is chosen such that it is consistent with all of $x_{\leq k}$.

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# Consistent Agent (cont'd)

*Recall we are assuming observations $x_k$, $k \in \mathbb{N}$ to be i.i.d.!*

## Lemma 2

*For the hypothesis $h_{k+1}$ of a $\mathcal{H}$-consistent agent satisfying* (5), $\mathrm{err}(h_{k+1}) \leq \epsilon$ *with probability at least* $1 - \delta$ *if*

$$k \geq \frac{1}{\epsilon} \ln \frac{|\mathcal{H}|}{\delta} \tag{6}$$

Proof: Due to the assumption (5), $\mathcal{H}$ contains a hypothesis consistent with an arbitrary sequence of observations $x_{\leq k}$, so the agent can pick one at each $k+1$ and the agent has at most $|\mathcal{H}|$ possible choices. The probability that one such chosen hypothesis $h_{k+1} \in \mathcal{H}$ consistent with $x_{\leq k}$ is bad ($\mathrm{err}(h_{k+1}) > \epsilon$) is $(1 - \mathrm{err}(h_{k+1}))^k < (1 - \epsilon)^k$.

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# Consistent Learning with with Polynomial lg $|\mathcal{H}|$

The probability that *some* of the $|\mathcal{H}|$ possible choices is bad is thus at most

$$|\mathcal{H}|(1-\epsilon)^k < |\mathcal{H}|e^{-\epsilon k}$$

where we used the bound (3). This is smaller than $\delta$ if (6) holds. This completes the proof. We get the following theorem as a corollary.

## Theorem 5

*An $\mathcal{H}$-consistent learning agent satisfying (5) PAC-learns from $X$ any concept class $\mathcal{C}$ on $X$ if* $\lg |\mathcal{H}| \leq poly(n_X)$.

Proof: By Lemma (2), an $\mathcal{H}$-consistent learning agent satisfying (5) has $\mathrm{err}(h_k) \leq \epsilon$ with probability at least $1-\delta$ for $k$ polynomial in $\frac{1}{\epsilon}, \frac{1}{\delta}$ (6). The requirement $\lg |\mathcal{H}| \leq poly(n_X)$ means that $k$ is also polynomial in $poly(n_X)$, thus the agent PAC-learns $\mathcal{C}$.

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# Consistent Learning with with Polynomial VC($\mathcal{H}$)

## Lemma 3

*For the hypothesis $h_{k+1}$ of a $\mathcal{H}$-consistent agent satisfying* (5), $\mathrm{err}(h_{k+1}) \leq \epsilon$ *with probability at least* $1 - \delta$ *if*

$$k \geq \max\left\{\frac{4}{\epsilon}\lg\frac{2}{\delta}, \frac{8 \cdot \mathrm{VC}(\mathcal{H})}{\epsilon}\lg\frac{13}{\epsilon}\right\} \tag{7}$$

Proof omitted due to length and technicality.

*$\mathrm{VC}(\mathcal{H})$ may be finite and polynomial even when $|\mathcal{H}|$ is infinite and $\lg|\mathcal{H}|$ is not defined. Even for a finite $|\mathcal{H}|$,* (7) *may give a better (smaller) bound than* (6). *(Exercise problem)*

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# Consistent Learning with with Polynomial VC($\mathcal{H}$) (cont'd)

A corollary of Lemma (3) is analogical to Theorem (5).

**Theorem 6**

*An $\mathcal{H}$-consistent learning agent satisfying (5) PAC-learns from $X$ any concept class $\mathcal{C}$ on $X$ if $\mathrm{VC}|\mathcal{H}| \leq poly(n_X)$.*

The proof is also analogical to that of Theorem (5), except it relies on Lemma (3) instead of Lemma (2).

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