Classification

Classification is a special case of the agent-environment interaction defined by two assumptions

1. Y is finite
2. rewards are instant (??)

Elements of Y are called *classes*.

Similarly, for 'regression' we would replace the first condition with $Y = \mathbb{R}$. We do not elaborate regression in this course.

(exercise problem)

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Example: Classification of Handwritten Numbers

$X = \mathbb{R}^{16\times 16}$ (x are pixel vectors)
$Y = \{\, 0, \ldots, 9 \,\}$

When training a classification agent, if we know the 'true' classes $\overline{y}(x)$ of observations, we may use them to prescribe rewards by function $r : Y \times Y \to R$, such as

$$r_{k+1} = r(y_{k+1}, \overline{y}(x_k)) = -|y_{k+1} - \overline{y}(x_k)|$$

$$r_{k+1} = r(y_{k+1}, y^*(x_k)) = \begin{cases} 0 \text{ if } y_{k+1} = \overline{y}(x_k) \\ -1 \text{ otherwise} \end{cases} \quad \text{(Unit reward)} \qquad (1)$$

The negative reward function $-r(.,.)$ is called a **loss function**.

Concept Classification

We will now focus on the simplest interesting form of classification: only two classes and no "noise".

Formally, any subset $C \subseteq X$ is called a **concept on X** and we define **concept classification** as a special case of classification where $Y = \{0, 1\}$ there is a **target concept** C on X instantiating the rewards (??) as ($k \in \mathbb{N}$)

$$r_1 = 0$$

$$r_{k+1} = \begin{cases} y_{k+1} - 1 \text{ if } x_k \in C \\ -y_{k+1} \text{ otherwise} \end{cases} \tag{2}$$

Concept Classification (cont'd)

In other words, the agent decides by $y_{k+1} = 1$ ($y_{k+1} = 0$) that $x_k \in C$ ($x_k \notin C$, respectively) and gets reward $r_{k+1} = 0$ ($r_{k+1} = -1$) if the decision was right (wrong, respectively).

(Exercise problem)

Rewards r_{k+1} here depend deterministically on x_k and y_{k+1}, hence no "noise".

Note that arbitrary classification can be done by a finite number of concept classification agents. Indeed, since Y is finite, each $y \in Y$ can be represented by a binary number with $n \approx \lg |Y|$ digits. So we just let the target concept for the i's agent contain all $x \in X$ for which the i's digit of optimal action $\overline{y}$ is 1. This agent will learn to predict the i's digit of the optimal action for x.

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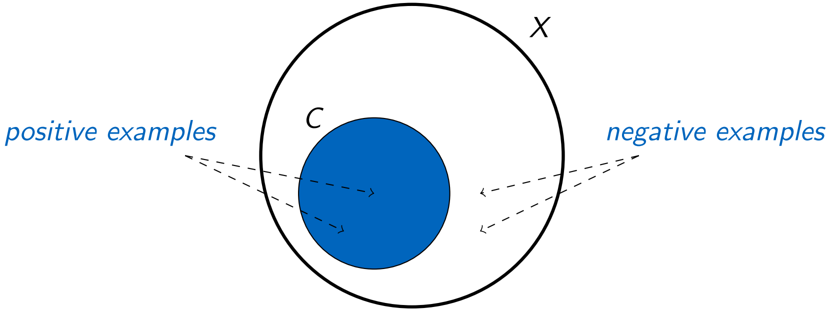
Positive and Negative Examples

Observations $x \in C$ ($x \notin C$, respectively) received by the agent are called **positive (negative) examples** of C.



Example: $X \sim$ descriptons of animals. $C \sim$ same for mammals. Positive example: description of a cat, negative example: same for a chicken.

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Hypothesis

From the examples, the agent learns a **hypothesis** h, which is a *finite-size* description of a binary policy. The hypothesis also induces a concept

$$C(h) = \{ x \in X \mid h(x) = 1 \} \tag{3}$$

Note that we overload the h symbol to denote both the description of the hypothesis and the policy function it defines.

(3) depends on the way the function $h(x)$ is determined from the description h. We do not make this dependence explicit as we will only be interested in hypotheses with an obvious functional interpretation.

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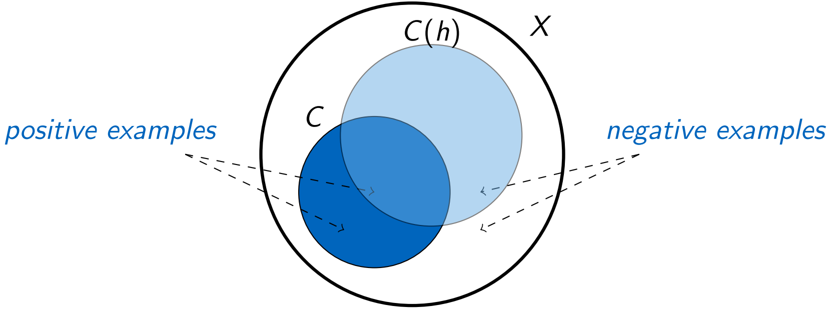
Hypothesis vs. Concept

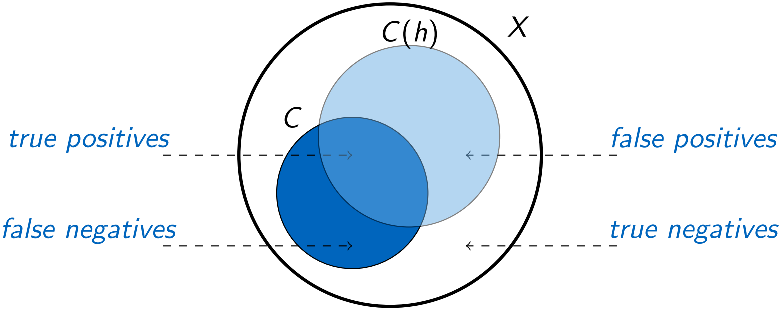
The goal of learning is to find h such that $C(h) = C$.



Example (logical): $h = \mathrm{milk} \wedge \neg\mathrm{feathers}$, $h(x) = 1$ iff $x \models h$.

Error Types

Until $C(h) = C$, there are four kinds of observations



False positives and false negatives form the *error region*.

Concept Class

With an unlimited supply of non-repeated examples, can we always learn the target concept, i.e. find a h such that

$$C(h) = C \tag{4}$$

In general, *no*. There are $2^{|X|}$ possible concepts on X. If X is infinite (e.g. $X = \mathbb{N}$), there is an uncountable ($|\mathbb{R}|$) number of such concepts. A hypothesis is a finite description so there is a countable number ($|\mathbb{N}|$) of hypotheses. Thus there are more concepts than hypotheses.

To allow any learnability results, we will always have to assume that C is not arbitrary ($C \in 2^X$) but belongs to a smaller **concept class on X**

$$\mathcal{C} \subset 2^X \tag{5}$$

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Hypothesis Class

A **hypothesis class** $\mathcal{H}$ is a set of hypotheses. For example, a set of *propositional-logic conjunctions*.

$\mathcal{C}(\mathcal{H})$ denotes the concept class on X induced by hypotheses in $\mathcal{H}$, i.e.

$$\mathcal{C}(\mathcal{H}) = \{ \, C(h) \mid h \in \mathcal{H} \, \} \tag{6}$$

So if $\mathcal{H}$ = propositional-logic conjunctions then $\mathcal{C}(\mathcal{H})$ means the set of all concepts on X that can be described by such conjunctions.

Terminology: when there is no risk of confusion, we will call $\mathcal{C}(\mathcal{H})$ the same as $\mathcal{H}$, e.g. "conjunctions " rather than "concept class on X induced by conjuctions". If $C(\overline{h})$ is the target concept, we will call $\overline{h}$ the **target hypothesis**.

Mistake-Bound Learning Model

Given a concept class $\mathcal{C}$ we want to study whether a learning agent can learn concepts from $\mathcal{C}$. What does "can learn concepts from $\mathcal{C}$" exactly mean? One definition is provided by the **mistake-bound** learning model also known as the **online learning** model.

Due to (2), maximizing the utilities (??) or (??) means minimizing the number of mistakes, i.e., actions followed immediately by reward $r = -1$. But what utility value is considered a success?

In the mistake-bound model, we request that if the target concept $C \in \mathcal{C}$, the number of mistakes is *finite* even for an infinite time horizon m, and this is true for *any distribution of observations* (??).

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Mistake-Bound Learning Model (cont'd)

Given (2), the total number of mistakes in one possible history $xr_{\leq k}$ is $\sum_{k=1}^{\infty} |r_k|$. Since the latter must be finite, $\sum_{k=1}^{\infty} r\gamma^k$ converges even with $\gamma = 1$ (we will keep $\gamma = 1$ unless stated otherwise).

As this must be true for any distribution of observations (??), the expectation in the infinite utility (??) also converges and $|U^{y_1, y_2, \cdots}|$ is the expected total number of mistakes.

Recall that the sequence of observations determined by distribution (??) is the only source of randomness in the agent-environment interaction in concept classification as (??) is set deterministically by (2).

Mistake-Bound Learning Model (cont'd)

Moreover, the model requests that the number of mistakes is not just finite, but reasonably small. In particular, it should grow at most *polynomially* with the *size (descriptive complexity)* of observations $x \in X$, denoted n_X. When observations are feature tuples of dimension n, we will always set $n_X = n$.

The model also defines when concept learning is time-efficient.

Mistake-bound model (or Online learning model)

In the concept classification protocol, an agent **learns $\mathcal{C}$ from X online** if for any target concept from $\mathcal{C}$ on X and an arbitrary distribution (??), it makes a sequence of decisions $y_1, y_2, \ldots$ such that $\sum_{k=1}^{\infty} |r_k| \leq poly(n_X)$. It learns $\mathcal{C}$ online from X **efficiently**, if in addition, the time taken to compute an action from an observation is also at most polynomial in n_X.

(*Exercise problem*)

The Winnow Algorithm

Winnow assumes Boolean-tuple observations, i.e. $X = \{0, 1\}^n, n \in \mathbb{N}$ and tries to identify the target concept $C \subseteq X$ by a hyperplane in X.

The agent's hypothesis at time k is an n-tuple of integers $h_k = (h_k^1, h_k^2, \ldots, h_k^n)$ specifying the hyperplane, initially set to

$$h_1 = (1, 1, \ldots, 1) \tag{7}$$

Its policy $y_{k+1} = h_k(x_k)$ $(k \in \mathbb{N})$ is given by

$$h_k(x_k) = 1 \text{ iff } \sum_{i=1}^{n} h_k^i x_k^i > \frac{n}{2} \tag{8}$$

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The Winnow Algorithm (cont'd)

On each mistake, h_k is updated to h_{k+1} with a simple learning rule:

- On a false negative x_k ($y_{k+1} = 0$, $r_{k+1} = -1$), *promote* each component h_k^i where $x_k^i = 1$ by doubling its value:

$$h_{k+1}^i = 2h_k^i \tag{9}$$

- On a false positive ($y_{k+1} = 1$, $r_{k+1} = -1$), *eliminate* each component h_k^i where $x_k^i = 1$ by zeroing it:

$$h_{k+1}^i = 0 \tag{10}$$

Winnow Learning Monotone Disjunctions

Using hyperplane separation, Winnow can learn only classes of linearly separable concepts. One example is the class $\mathcal{C}(\mathcal{H})$ of *monotone disjunctions* made out of up to n propositional variables $\mathrm{p}_1, \mathrm{p}_2, \ldots \mathrm{p}_n$, i.e.,

$$\mathcal{H} = \{ \mathrm{p}_{i_1} \vee \mathrm{p}_{i_2} \vee \ldots \mathrm{p}_{i_s} \mid 1 \leq i_j \leq n \}$$

So, for example, when $n = 4$ and the target hypothesis is $\mathrm{p}_1 \vee \mathrm{p}_3$, the agent's hypothesis $h = [2, 0, 1, 0]$ will not make any mistakes because the target disjunction is true iff

$$2x^1 + x^3 > 2$$

We are putting component indexes to the superscript of x and h_k, reserving the subscript for a time index.

Conjunctive Observations

For Winnow, we assumed $X = \{0, 1\}^n$, $n \in \mathbb{N}$, so an example $x \in X$ specifies *each* of the n Boolean values. We want to allow the case that x does not specify some of them. This could be done by leting $X = \{0, 1, ?\}^n$ where ? stands for *value uknown*.

Another way is to let X be the set of **contingent** (not tautologically true or false) *conjunctions* of propositional literals made of atoms selected from $\mathrm{p}_1, \mathrm{p}_2, \ldots, \mathrm{p}_n$. For example, with $n = 3$ the observation

$$\mathrm{p}_1 \wedge \neg \mathrm{p}_3$$

represents the same information as

$$(1, ?, 0)$$

We define the complexity n_X of such observations to be n.

(*Exercise problem*)

Conjunctive Hypotheses

Unless stated otherwise, the term conjunction (disjunction) will mean a conjunction (disjunction) of *propositional literals*, excluding e.g. a conjunction of disjunctions (or the reverse). Non-propositional cases will be marked explicitly.

We will be interested in hypotheses which are conjunctions, providing the following decision policy $y = h(x)$ for conjunctive examples

$$h(x) = 1 \text{ iff } x \models h \tag{11}$$

where $\models$ is *tautological consequence*. So e.g. the observation

$$x = \mathrm{milk} \wedge \neg \mathrm{feathers} \wedge \neg \mathrm{flies}$$

is decided positively ($h(x) = 1$) by $h = \mathrm{milk} \wedge \neg \mathrm{feathers}$.

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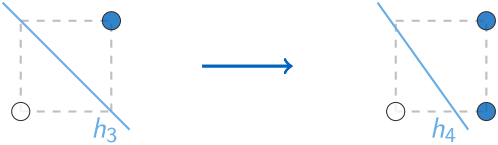
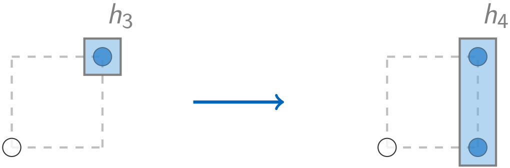
Separation vs. Generalization

Winnow uses the popular learning technique of *separation*



An alternative is the "covering" approach seeking the smallest joint *generalization* of positive examples



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Generality and Subsumption Order

Let π, π' be two policies $X \to \{0, 1\}^n$. We say that y is **at least as general as** y' if $\pi(x) = 1$ for any $x \in X$ such that $\pi'(x) = 1$.

Let h, h' be conjunctions that prescribe policies by (11). If $h' \models h$ then h is at least as general as h'. (*exercise problem*)

Let h, h' be two conjunctions or two disjunctions. We say that h **subsumes** h' (written $h \subseteq h'$) if $\mathrm{Lits}(h) \subseteq \mathrm{Lits}(h')$ where $\mathrm{Lits}(c)$ denotes the set of literals in c. We say that h **strictly subsumes** (written $h \subset h'$) h' if $\mathrm{Lits}(h) \subset \mathrm{Lits}(h')$.

Theorem 1

Let h, h' be conjunctions. If $h \subseteq h'$ then $h' \models h$. Let furthermore h' not be tautologically false. Then $h \subseteq h'$ if and only if $h' \models h$.

(*exercise problem*)

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Least General Generalization

Let h, h' be two conjunctions (two disjunctions, respectively). We say that g is a **least general generalization** of h and h' if $g \subseteq h$, $g \subseteq h'$, and there is no conjunction (disjunction) g' such that $g \subset g'$, $g' \subseteq h$, $g' \subseteq h'$.

Let h, h' be two conjunctions (two disjunctions, respectively) and let us define:

$$\text{lgg}(\mathbf{h}, \mathbf{h'}) = \text{Lits}(h) \cap \text{Lits}(h') \tag{12}$$

Easy to verify: $\text{lgg}(h, h')$ is a least general generalization of h and h'.

The proof is an exercise problem.

(*a further exercise problem*)

LGG Properties and Set-LGG

The subsumption order $\subseteq$ induces a *lattice* where lgg is the *least upper bound* (lup). As any lup, lgg has these properties:

$$\text{lgg}(a, b) = a \text{ if } a \subseteq b \tag{13}$$

$$\text{lgg}(a, b) = \text{lgg}(b, a) \text{ (commutativity)} \tag{14}$$

$$\text{lgg}(a, \text{lgg}(b, c)) = \text{lgg}(\text{lgg}(a, b), c) \text{ (associativity)} \tag{15}$$

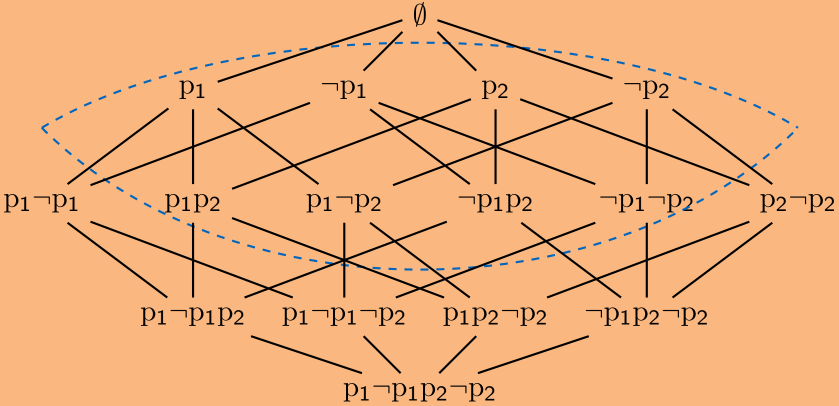
Properties 14 and 15 let us extend lgg naturally to *sets* of conjunctions or disjunctions:

$$\text{lgg}(\{ x_1, x_2, \ldots, x_m \}) = \text{lgg}(\ldots \text{lgg}(\text{lgg}(x_1, x_2), \ldots), x_m) \tag{16}$$

where the order of the x_k on the RHS is irrelevant. For conjunctions, obviously $\text{lgg}(\{ x_1, x_2, \ldots, x_m \}) = \bigcap_{k=1}^{m} x_k$.

Example: Subsumption Lattice on Conjunctions

Contingent conjunctions are enclosed by the dashed curve.



The Generalization Algorithm

The generalization algorithm assumes X to consist of contingent conjunctions on variables $\mathrm{p}_1, \mathrm{p}_2, \ldots, \mathrm{p}_n$. It uses the policy (11), which can be written as

$$h(x) = 1 \text{ if } h \subseteq x \tag{17}$$

because $x \in X$ are contingent. It has a simple learning rule:

- Wait for the first positive example. That is, emit actions $y_k = 0$ until $r_k = -1$, then x_{k-1} is a positive example. Set $h_k = x_{k-1}$.
- Continue receiving percepts and on each mistake ($r_{k+1} = -1$), set

$$h_{k+1} = \mathrm{lgg}(h_k, x_k) \tag{18}$$

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