# GVG Test 3
# Solution

May 22, 2021

**Task 1.** *Let us have a mapping* $\varphi\colon \mathbb{R}^3 \to \mathbb{R}$ *such that*

$$\varphi\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = 1, \quad \varphi\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = 2$$

*Define*

$$\varphi\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) =$$

*such that* $\varphi$ *can be linear mapping. Justify.*

**Solution:** For the linear mapping there must hold true

$$\varphi(\mathtt{x} - \mathtt{y}) = \varphi(\mathtt{x}) - \varphi(\mathtt{y})$$

Thus,

$$\varphi\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \varphi\left(\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \varphi\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) - \varphi\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = 1 - 2 = -1.$$

□

**Task 2.** *Let us have a point* $\vec{u}_\alpha = [0,0]^\top$ *in an image captured by camera with matrix*

$$\mathtt{K} = \begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 1\end{bmatrix}$$

*Write the affine coordinates of point* $\vec{v}_\alpha$ *such that the rays of points* $u$ *and* $v$ *form a* $45^\circ$ *angle.*

**Solution:** Let's denote the directions of the two rays by $\vec{x}_1$ and $\vec{x}_2$. By the task, $\vec{x}_{1\beta} = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^\top$ and thus,

$$\vec{x}_{1\gamma} = \mathtt{K}^{-1}\vec{x}_{1\beta} = \begin{bmatrix}\frac{1}{2} & 0 & 0\\0 & \frac{1}{2} & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

We know that

$$\frac{\sqrt{2}}{2} = \cos 45^\circ = \cos\angle(\vec{x}_1, \vec{x}_2) = \frac{\vec{x}_{1\gamma}^\top \vec{x}_{2\gamma}}{\|\vec{x}_{1\gamma}\|\,\|\vec{x}_{2\gamma}\|} = \frac{\begin{bmatrix}0 & 0 & 1\end{bmatrix}\vec{x}_{2\gamma}}{\left\|\begin{bmatrix}0\\0\\1\end{bmatrix}\right\|\|\vec{x}_{2\gamma}\|} = \frac{\vec{x}_{2\gamma}^{(3)}}{\|\vec{x}_{2\gamma}\|}$$

There are infinitely many possible $\vec{x}_{2\gamma}$ which satisfy the above equation. One of them is, for example,

$$\vec{x}_{2\gamma} = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

To write the affine coordinates of $\vec{v}$ we need to transform $\vec{x}_{2\gamma}$ to basis $\beta$:

$$\vec{x}_{2\beta} = \mathtt{K}\vec{x}_{2\gamma} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

Hence

$$\vec{v}_\alpha = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

□

**Task 3.** *Let us have two images with projection matrices and epipole*

$$\mathtt{P}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathtt{P}_2 = \begin{bmatrix} 1 & 0 & 0 & -x \\ 0 & 1 & 0 & -y \\ 0 & 0 & 1 & -z \end{bmatrix}, \quad \vec{e}_\alpha = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

*in the first image. Find all camera centers $\vec{C}_{2\delta}$ of the second camera compatible with the given arrangement.*

**Solution:** By definition, the epipole in the first image is the projection of the center of the second camera to the first camera:

$$\zeta_1 \vec{e}_{1\beta_1} = \mathtt{P}_1 \begin{bmatrix} \vec{C}_{2\delta} \\ 1 \end{bmatrix}, \quad \zeta_1 \neq 0.$$

We can notice that the left $3 \times 3$ block of $\mathtt{P}_2$ is the identity matrix and thus $\vec{C}_{2\delta} = -\mathtt{P}_{:,4} = \begin{bmatrix} x & y & z \end{bmatrix}^\top$. After substituting the known epipole and the unknowns for $\vec{C}_{2\delta}$ we obtain

$$\zeta_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mathtt{P}_1 \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ z+1 \\ y \end{bmatrix}, \quad \zeta_1 \neq 0.$$

Therefore, all the possible centers of the second camera are

$$\left\{ \begin{bmatrix} \zeta_1 \\ \zeta_1 \\ \zeta_1 - 1 \end{bmatrix} \;\middle|\; \zeta_1 \in \mathbb{R}\backslash\{0\} \right\}.$$

□

**Task 4.** *Consider two cameras with projection matrices*

$$\mathtt{P}_1 = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathtt{P}_2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

*Find point $\vec{X}_\delta$ in space that projects into image points $\vec{u}_{1\alpha_1} = [\frac{1}{2}, 0]^\top$, $\vec{u}_{2\alpha_2} = [3, 0]^\top$.*

**Solution:** We have the following equations:

$$\zeta_1 \vec{x}_{1\beta} = \mathtt{P}_1 \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}, \quad \zeta_2 \vec{x}_{2\beta} = \mathtt{P}_2 \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}$$

By the task, $\vec{x}_{1\beta} = \begin{bmatrix} \frac{1}{2} & 0 & 1 \end{bmatrix}^\top$ and $\vec{x}_{2\beta} = \begin{bmatrix} 3 & 0 & 1 \end{bmatrix}^\top$. Substituting all the known values to the above equations and denoting $\vec{X}_\delta = \begin{bmatrix} x & y & z \end{bmatrix}^\top$ we get

$$\zeta_1 \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, \quad \zeta_2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

which is equivalent to

$$\zeta_1 \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -x+1 \\ -y+1 \\ z \end{bmatrix}, \quad \zeta_2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} z+1 \\ y-1 \\ x+1 \end{bmatrix}$$

Expressing $x, y, z$ from both matricial equations we get

$$x = -\frac{1}{2}\zeta_1 + 1, \quad y = 1, \quad z = \zeta_1,$$

$$x = \zeta_2 - 1, \quad y = 1, \quad z = 3\zeta_2 - 1.$$

Making the expressions for $x, y, z$ equal we obtain

$$-\frac{1}{2}\zeta_1 + 1 = \zeta_2 - 1, \quad \zeta_1 = 3\zeta_2 - 1$$

$$-\frac{1}{2}(3\zeta_2 - 1) + 1 = \zeta_2 - 1 \Rightarrow \zeta_2 = 1 \Rightarrow \zeta_1 = 3 \cdot 1 - 1 = 2$$

Thus,

$$x = -\frac{1}{2} \cdot 2 + 1 = 0, \quad y = 1, \quad z = 2,$$

and

$$\vec{X}_\delta = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

□

**Task 5.** *Suppose we are given 2 calibrated cameras and the calibrated camera projection matrix of the first camera is*

$$\mathtt{P}_{1\gamma_1} = \begin{bmatrix} \mathtt{R}_1 & -\mathtt{R}_1\vec{C}_{1\delta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

*Find all possible centers $\vec{C}_{2\delta}$ of the second camera knowing that a multiple of the essential matrix which relates these cameras is*

$$\mathtt{G} = \tau\mathtt{E} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \tau \neq 0.$$

**Solution:** By definition, the essential matrix takes the form

$$\mathtt{E} = \mathtt{R}_2 \left[\vec{C}_{2\delta} - \vec{C}_{1\delta}\right]_\times \mathtt{R}_1^\top$$

Hence

$$\mathtt{G} = \tau\mathtt{E} = \tau\mathtt{R}_2 \left[\vec{C}_{2\delta} - \vec{C}_{1\delta}\right]_\times \mathtt{R}_1^\top$$

$$\mathtt{G}\mathtt{R}_1 = \tau\mathtt{R}_2 \left[\vec{C}_{2\delta} - \vec{C}_{1\delta}\right]_\times$$

Since $\vec{C}_{1\delta} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top$, then after substituting everything we are given by the task we obtain

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \tau\mathtt{R}_2 \left[\vec{C}_{2\delta}\right]_\times$$

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathtt{A}} = \tau\mathtt{R}_2 \left[\vec{C}_{2\delta}\right]_\times \tag{1}$$

We claim that all the solutions $\vec{C}_{2\delta}$ can be obtained as the kernel of $\mathtt{A}$ (from which we must exclude $\vec{C}_{2\delta} = \vec{C}_{1\delta} = \mathtt{0}$), i.e. $S = \ker \mathtt{A} \backslash \{\mathtt{0}\}$, where $S$ is the set of solutions for $\vec{C}_{2\delta}$. We immediately see that $\vec{C}_{2\delta}$ can't be the zero vector, because otherwise the right-hand side of Equation (1) would become zero, but the left-hand side is not. We also see that any solution $\vec{C}_{2\delta}$ to the Equation (1) belongs to $\ker \mathtt{A}$, since

$$\mathtt{A} = \tau \mathtt{R}_2 \left[\vec{C}_{2\delta}\right]_\times \Rightarrow \mathtt{A}\vec{C}_{2\delta} = \tau \mathtt{R}_2 \left[\vec{C}_{2\delta}\right]_\times \vec{C}_{2\delta} = \mathtt{0} \tag{2}$$

Finally, we need to show that any element from $\ker \mathtt{A} \backslash \{\mathtt{0}\}$ is a solution to Equation (1). By the task we know that Equation (1) (with unknowns $\tau, \mathtt{R}_2, \vec{C}_{2\delta}$) has at least one solution $(\tau^*, \mathtt{R}_2^*, \vec{C}_{2\delta}^*)$ (because when we say that $\mathtt{G}$ relates the 2 cameras, it automatically means, that such a second camera exists), i.e.

$$\mathtt{A} = \tau^* \mathtt{R}_2^* \left[\vec{C}_{2\delta}^*\right]_\times \tag{3}$$

By the argument from Equation (2) we know that $\vec{C}_{2\delta}^* \in \ker \mathtt{A} \backslash \{\mathtt{0}\}$. Thus, any element from $\ker \mathtt{A} \backslash \{\mathtt{0}\}$ can be obtained as $\sigma \vec{C}_{2\delta}^*$ for a nonzero $\sigma \in \mathbb{R}$. We set the new solution to be $(\frac{\tau^*}{\sigma}, \mathtt{R}_2^*, \sigma \vec{C}_{2\delta}^*)$ and we check if it indeed the solution to Equation (1):

$$\mathtt{A} = \frac{\tau^*}{\sigma} \mathtt{R}_2^* \left[\sigma \vec{C}_{2\delta}^*\right]_\times = \frac{\tau^*}{\sigma} \mathtt{R}_2^* \sigma \left[\vec{C}_{2\delta}^*\right]_\times = \tau^* \mathtt{R}_2^* \left[\vec{C}_{2\delta}^*\right]_\times$$

which holds true due to Equation (3). We find the kernel of $\mathtt{A}$:

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using Gaussian Elimination we can solve this system and obtain the set of solutions:

$$S = \ker \mathtt{A} \backslash \{\mathtt{0}\} = \left\{ \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \backslash \{0\} \right\}.$$

□

**Remark 1.** *We could use another strategy to solve this task. We could find the calibrated epipole in the first camera:*

$$\mathtt{G}\vec{e}_{1\gamma_1} = \tau \mathtt{E} \vec{e}_{1\gamma_1} = \mathtt{0} \Rightarrow \vec{e}_{1\gamma_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

*Now, we know that*

$$\lambda \vec{e}_{1\gamma_1} = \mathtt{P}_{1\gamma_1} \begin{bmatrix} \vec{C}_{2\delta} \\ 1 \end{bmatrix}, \quad \lambda \neq 0$$

*from which we obtain the set of solutions to $\vec{C}_{2\delta}$:*

$$\left\{ \begin{bmatrix} \lambda \\ \lambda \\ 0 \end{bmatrix} \;\middle|\; \lambda \in \mathbb{R} \backslash \{0\} \right\}.$$

**Remark 2.** *Notice that the kernel of $\mathtt{G}$ is not $\vec{C}_{2\delta}$, but $\mathtt{R}_1(\vec{C}_{2\delta} - \vec{C}_{1\delta})$, since*

$$\mathtt{G}\mathtt{R}_1(\vec{C}_{2\delta} - \vec{C}_{1\delta}) = \tau \mathtt{R}_2 \left[\vec{C}_{2\delta} - \vec{C}_{1\delta}\right]_\times \underbrace{\mathtt{R}_1^\top \mathtt{R}_1}_{\mathtt{I}} (\vec{C}_{2\delta} - \vec{C}_{1\delta}) = \tau \mathtt{R}_2 \left[\vec{C}_{2\delta} - \vec{C}_{1\delta}\right]_\times (\vec{C}_{2\delta} - \vec{C}_{1\delta}) = \mathtt{0}.$$

*It would be equal to $\vec{C}_{2\delta}$ if $\mathtt{R}_1$ was equal to $\mathtt{I}$. This is actually what we did in slide 3/8 in [1]: we first transformed the two cameras by*

$$\mathtt{H}^{-1} = \begin{bmatrix} \mathtt{R}_1^\top & \vec{C}_{1\delta} \\ \mathtt{0}^\top & 1 \end{bmatrix}$$

*After this transformation we get the set of new cameras $(\mathtt{I}, \mathtt{0})$ and $(\mathtt{R}_2', \vec{C}_{2\delta}') = (\mathtt{R}_2 \mathtt{R}_1^\top, \mathtt{R}_1(\vec{C}_{2\delta} - \vec{C}_{1\delta}))$ and*

$$\mathtt{G} = \tau \mathtt{E} = \tau \mathtt{R}_2' \left[\vec{C}_{2\delta}'\right]_\times \Rightarrow \ker \mathtt{G} = \langle \vec{C}_{2\delta}' \rangle.$$

*To get, however, $\vec{C}_{2\delta}$ from $\vec{C}_{2\delta}'$ we need to do the following:*

$$\vec{C}_{2\delta}' = \mathtt{R}_1(\vec{C}_{2\delta} - \vec{C}_{1\delta}) \Rightarrow \vec{C}_{2\delta} = \mathtt{R}_1 \vec{C}_{2\delta}' + \vec{C}_{1\delta}.$$

## References

[1] *Gvg, lab 12*, https://cw.fel.cvut.cz/b202/_media/courses/gvg/labs/gvg_lab12.pdf.