1. Let us have coordinates $[u, v]^\top$ of the points in the image. Write the three-dimensional coordinates of points in space, which are projected on a line $u = v$ by camera matrix

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

- LINE $u = v \longrightarrow \vec{u} = \begin{bmatrix} u \\ u \\ 1 \end{bmatrix}$

$$\lambda \vec{u} = P\vec{X} \; ; \; \vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}, \lambda \neq 0$$

$$\begin{bmatrix} \lambda u \\ \lambda u \\ \lambda \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ x_1 + x_2 \\ x_3 + 2 \end{bmatrix} \longrightarrow \begin{array}{l} x_1 = \lambda u - \lambda + 2 \\ x_2 = \lambda - 2 \\ x_3 = \lambda - 2 \end{array} \longrightarrow \vec{X} = \begin{bmatrix} \lambda u - \lambda + 2 \\ \lambda - 2 \\ \lambda - 2 \end{bmatrix} \quad \begin{array}{l} \lambda \in \mathbb{R} \setminus \{0\} \\ u \in \mathbb{R} \end{array}$$

2. Let us have a homography given by matrix

$$\mathtt{H} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

which projects plane π on plane π'.

(a) Find the image of the ideal line in π.

(b) Find the preimage of the ideal line in π'.

– IDEAL LINE IN π: $\vec{l} = \begin{bmatrix} 0 \\ 0 \\ l \end{bmatrix}$; $l \neq 0$

– IDEAL LINE IN π': $\vec{k}' = \begin{bmatrix} 0 \\ 0 \\ k' \end{bmatrix}$; $k \neq 0$

a)

$\lambda \vec{l}' = H^{-T} \vec{l}$; $\lambda \neq 0$

$\lambda \vec{l}' = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ l \end{bmatrix} = \begin{bmatrix} -l \\ -l \\ l \end{bmatrix}$; $a = \frac{l}{\lambda} \longrightarrow \vec{l}' = \begin{bmatrix} -a \\ -a \\ a \end{bmatrix}$; $a \neq 0$

b)

$\rho \vec{k}' = H^{-T} \vec{k} \rightarrow \vec{k} = \rho H^{T} \vec{k}'$; $\rho \neq 0$

$\vec{k} = \rho \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ k' \end{bmatrix} = \rho \begin{bmatrix} k' \\ k' \\ k' \end{bmatrix}$; $b = \rho k' \rightarrow \vec{k} = \begin{bmatrix} b \\ b \\ b \end{bmatrix}$; $b \neq 0$

3. Points in an affine plane with affine coordinates

$$\vec{X}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \vec{X}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \vec{X}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \vec{X}_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

are mapped by homography into image points with affine coordinates

$$\vec{u}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \vec{u}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \vec{u}_3 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \ \vec{u}_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(a) Calculate a homography matrix.

(b) Find the affine coordinates of the point in the plane which maps into point $[1, 2]^\top$ in the image.

$$\lambda \vec{u} = H\vec{X}$$

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \rightarrow \begin{aligned} \lambda u &= h_{11}x + h_{12}y + h_{13} \\ \lambda v &= h_{21}x + h_{22}y + h_{23} \\ \lambda &= h_{31}x + h_{32}y + h_{33} \end{aligned}$$

$$\boxed{\begin{aligned} h_{31}ux + h_{32}uy + h_{33}u - h_{11}x - h_{12}y - h_{13} &= 0 \\ h_{31}vx + h_{32}vy + h_{33}v - h_{21}x - h_{22}y - h_{23} &= 0 \end{aligned}}$$

(a) $i=1$: $u=0,\ v=0,\ x=0,\ y=0$

$\boxed{h_{13}=0}$

$\boxed{h_{23}=0}$

$i=2$: $u=1,\ v=0,\ x=1,\ y=0$

$\boxed{h_{31}+h_{33}-h_{11}=0}$ A

$\boxed{h_{21}=0}$

$i=3$: $u=0,\ v=2,\ x=0,\ y=1$

$\boxed{h_{12}=0}$

$\boxed{2h_{32}+2h_{33}-h_{22}=0}$ B

$i=4$: $u=1,\ v=1,\ x=1,\ y=1$

$\boxed{h_{31}+h_{32}+h_{33}-h_{11}=0}$

$\boxed{h_{31}+h_{32}+h_{33}-h_{22}=0}$ C

$\rightarrow \boxed{h_{11}=h_{22}=a}$

A $\quad h_{31}+h_{33}-a=0 \quad \rightarrow h_{31}=a-h_{33}$

B $\quad 2h_{32}+2h_{33}-a=0 \quad \rightarrow h_{32}=\frac{a}{2}-h_{33}$

C $\quad h_{31}+h_{32}+h_{33}-a=0 \quad \rightarrow a-h_{33}+\frac{a}{2}-h_{33}+h_{33}-a=0$

$\rightarrow \boxed{h_{33}=\frac{a}{2}}$

$\rightarrow \boxed{h_{32}=0}$

$\rightarrow \boxed{h_{31}=\frac{a}{2}}$

$$H = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ \frac{a}{2} & 0 & \frac{a}{2} \end{bmatrix}$$

(b) $$\lambda \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda \\ 2\lambda \\ \lambda \end{bmatrix} = H \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a x_1 \\ a x_2 \\ \frac{a}{2}x_1 + \frac{a}{2}x_3 \end{bmatrix} ; \ \lambda \neq 0$$

$x_1 = \frac{\lambda}{a}$

$x_2 = \frac{2\lambda}{a}$

$x_3 = \left(\lambda - \frac{\cancel{a}}{2}\frac{\lambda}{\cancel{a}}\right)\cdot\frac{2}{a} = \frac{\lambda}{\cancel{2}}\frac{\cancel{2}}{a} = \frac{\lambda}{a}$

$x = \frac{\lambda}{a} \ \rightarrow \ \vec{X} = \begin{bmatrix} x \\ 2x \\ x \end{bmatrix} ; \ x \in \mathbb{R} \setminus \{0\}$

4. Let us have a line $\vec{p} = [1, 0, 1]^\top$ and points $\vec{X} = [2, 2, 2]^\top$, $\vec{Y} = [1, 0, 0]^\top$ in real projective plane. Find the intersection of line $\vec{p}$ with a line passing through points $\vec{X}, \vec{Y}$.

$$\vec{q} = \vec{X} \times \vec{Y} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$$

$$\vec{\Pi} = \vec{p} \times \vec{q} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$$

5. Consider two cameras with projection matrices

$$P_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Find the homography between the images of points in the plane $z = 0$.

$$G_1 = \begin{bmatrix} \vec{p}^{\,1}_1 & \vec{p}^{\,2}_1 & \vec{p}^{\,4}_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} \vec{p}^{\,1}_2 & \vec{p}^{\,2}_2 & \vec{p}^{\,4}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$H = G_2 G_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$