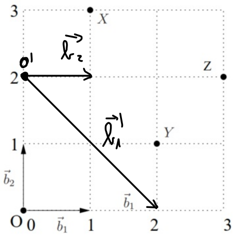
1. The following figure shows coordinate system $\sigma = (O, \beta)$, with basis $\beta = (\vec{b}_1, \vec{b}_2)$.



$X_\sigma = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$Y_\sigma = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$Z_\sigma = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

(a) Find the coordinate system $\sigma' = (O', \beta')$, $\beta' = (\vec{b}'_1, \vec{b}'_2)$, so that the points X, Y, Z have the following coordinates in σ'

$$X_{\sigma'} = \begin{bmatrix} -1/2 \\ 2 \end{bmatrix}, \quad Y_{\sigma'} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, \quad Z_{\sigma'} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

and plot the system into the figure.

(b) Write the coordinates of the vectors of β in basis β'.

(c) Write the coordinates of the point O in the coordinate system σ'.

(d) Write the coordinates of the vectors of β' in basis β.

$$\overrightarrow{O'O}_\beta$$
$$\|$$
$$\vec{X}_{\sigma'} = \beta_{\beta'} \vec{X}_\sigma + \vec{O}_{\sigma'}$$

$$\begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} o_1 \\ o_2 \end{bmatrix}$$

x_1: $-\frac{1}{2} = b_{11} + 3b_{21} + o_1 = b_{11} + 3b_{21} - 3b_{11} - 2b_{21} = -2b_{11} + 1b_{21} \rightarrow b_{21} = -\frac{1}{2} + 2b_{11}$

y_1: $\frac{1}{2} = 2b_{11} + b_{21} + o_1 = 2b_{11} + b_{21} - 3b_{11} - 2b_{21} = -b_{11} - b_{21} = -b_{11} + \frac{1}{2} - 2b_{11}$

z_1: $0 = 3b_{11} + 2b_{21} + o_1 \rightarrow o_1 = -3b_{11} - 2b_{21}$

$\Downarrow$

$\boxed{b_{11} = 0, \quad b_{21} = -\frac{1}{2}, \quad o_1 = 1}$

x_2: $2 = b_{12} + 3b_{22} + o_2 \rightarrow o_2 = 2 - b_{12} - 3b_{22}$

y_2: $1 = 2b_{12} + b_{22} + o_2 = 2b_{12} + b_{22} + 2 - b_{12} - 3b_{22} \rightarrow 1 + b_{12} - 2b_{22} = 0 \rightarrow b_{12} = -1 + 2b_{22}$

z_2: $3 = 3b_{12} + 2b_{22} + o_2 = 3b_{12} + 2b_{22} + 2 - b_{12} - 3b_{22} \rightarrow -1 + 2b_{12} - b_{22} = 0 = -1 - 2 + 4b_{22} - b_{22}$

$\Downarrow$

$\boxed{b_{22} = 1, \quad b_{12} = 1, \quad o_2 = -2}$

b) $\beta_{\beta'} = \begin{bmatrix} \vec{b}_{1\beta'} & \vec{b}_{2\beta'} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} \\ 1 & 1 \end{bmatrix}$

c) $\vec{O}_{\sigma'} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

d) $\beta'_\beta = \beta_{\beta'}^{-1} = \begin{bmatrix} \vec{b}'_{1\beta} & \vec{b}'_{2\beta} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}$ $\quad \vec{O}'_\sigma = \beta'_\beta(-\vec{O}_{\sigma'}) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

2. Assume a camera with the camera ~~projection~~ CALIBRATION matrix

$$\mathtt{K} = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find the focal length in pixels when knowing that the cosine of the angle between the rays passing through points $[1, -1]^\top$ a $[0, 1]^\top$ in the image is equal to 0.

$$\vec{x}_{i\beta} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad \vec{x}_{j\beta} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad K^{-1} = \begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/a & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$0 = \cos\sphericalangle(x_i, x_j) = \frac{\vec{x}_{i\beta}^{\top} K^{-T} K^{-1} \vec{x}_{j\beta}}{\|K^{-1}\vec{x}_{i\beta}\| \|K^{-1}\vec{x}_{j\beta}\|} = \frac{\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/a & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/a & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}{\|K^{-1}\vec{x}_{i\beta}\| \|K^{-1}\vec{x}_{j\beta}\|} = \frac{1 - \frac{1}{a^2}}{\|K^{-1}\vec{x}_{i\beta}\| \|K^{-1}\vec{x}_{j\beta}\|} \longrightarrow a = \pm 1$$

$$k_{11} = k_{22} = a \; ; \; k_{11} > 0 \longrightarrow k_{11} = k_{22} = +1$$

$$k_{11} = \frac{f}{\|\vec{k}_1\|} = +1 \quad [\mathrm{PX}]$$

3. Find the centers of all cameras that project 3D point $[1,1,1]^\top$ into the image point $[2,1]^\top$, with the following partial knowledge of the camera projection matrix

$$\mathrm{P} = \left[\underbrace{\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{KR} \;\; \underbrace{\begin{matrix} a \\ b \\ c \end{matrix}}_{-KRC}\right], \quad \vec{X} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{m} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda \vec{m} = P\vec{X} \rightarrow \lambda \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2\lambda \\ \lambda \\ \lambda \end{bmatrix} = \begin{bmatrix} 2+a \\ 1+b \\ 1+c \end{bmatrix} \rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2\lambda - 2 \\ \lambda - 1 \\ \lambda - 1 \end{bmatrix} = -KR\vec{C} = -\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{C}$$

$$\rightarrow \begin{bmatrix} 2\lambda - 2 \\ \lambda - 1 \\ \lambda - 1 \end{bmatrix} = \begin{bmatrix} -c_1 - c_3 \\ -c_2 \\ -c_3 \end{bmatrix} \rightarrow \vec{C} = \begin{bmatrix} 1-\lambda \\ 1-\lambda \\ 1-\lambda \end{bmatrix} \rightarrow \begin{matrix} 1 = 1 - \lambda \\ \lambda \neq 0 \\ 1 \neq 1 \end{matrix} \rightarrow \vec{C} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

4. Assume a camera with the following image projection matrix

$$P_\beta = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}. = \left[K_\beta R \mid -K_\beta R \vec{C}_\delta \right]$$

Find the coordinates of the projection center.

$$\downarrow$$

$$-\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$-c_1 - c_2 - c_3 = 0 \longrightarrow c_2 + 1 - c_2 + c_2 + 1 = 0 \rightarrow c_2 = -2$$

$$-c_1 - c_2 = 1 \longrightarrow c_1 = -c_2 - 1 \qquad c_1 = 1$$

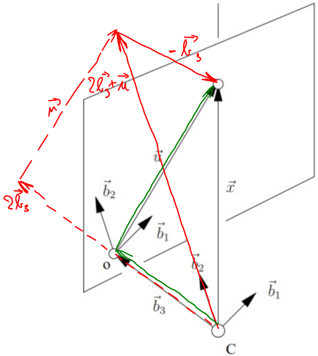
$$-c_2 - c_3 = 1 \longrightarrow c_3 = -c_2 - 1 \qquad c_3 = 1$$

$$\Downarrow$$

$$C_\delta = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$\alpha = (\vec{a}_1, \vec{a}_2, \vec{a}_3)$

5. Write down the coordinates of vector $\vec{x}$ in basis $(\vec{b}_3, \vec{b}_2, 2\vec{b}_3 + \vec{u})$ as shown in the following figure



$\vec{x} = \vec{b}_3 + \vec{u}$

$\vec{x} = 1(2\vec{b}_3 + \vec{u}) - 1\,\vec{b}_3$

$= 1\,\vec{a}_3 - 1\,\vec{a}_1$

$\vec{x}_\alpha = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$